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PASTING TOGETHER OF SPACE-TIME SLICES

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First, let me express my gratefulness to the Indian Association for General Relativity and Gravitation for inviting me to deliver the Vaidya—Raychaudhuri Endowment Award Lecture. I feel greatly honoured. I was a direct student of Late Prof. A.K. Raychaudhuri and look upon Prof. P.C. Vaidya as my teacher.

I shall describe today a useful mathematical tool that allows us to paste together two slices of space-time expressed in terms of different coordinate systems on the two sides of a 3-dimensional hyper-surface. This lecture may be useful to young workers in general relativity.

Matching conditions of space-time slices

Let us take a 3-space S dividing the space-time into two distinct four dimensional manifolds V^+ (interior space-time) and V^- (exterior space-time). If the same coordinate patch covers V^+ and V^- then we demand simply that the components of the metric tensor and their first derivatives be continuous across S. But we are going to describe a method which is independent of the coordinate system. We may cut out two slices of 4-spaces expressed in terms of different coordinate systems and paste them together on the two sides of a 3-space. The junction conditions then give the relations between the coordinates on the two sides. The method was originally given by Israel (1966, 1967).

The first condition for pasting is that the 3-space will have the same well-defined intrinsic geometry as viewed from the two sides. The two 4-spaces V^+ and V^- are supposed to be covered by the coordinate patches $y^a_\pm$ and the metrics are given by

$$ds^2_\pm = a_{ab}^\pm \, dy^a_\pm \, dy^b_\pm \qquad (1)$$

The Greek indices stand for 1, 2, 3, 4.

If g_{ij} be the intrinsic metric of the 3-space S covered by the coordinates x^i (latin indices represent 1, 2, 3), so that we have

$$ds^2_S = g_{ij} \, dx^i \, dx^j \qquad (2)$$

(2) is an invariant known as the *first fundamental form* [Weatherburn (1957) pp.123-129]. The 3-space S must have the same intrinsic geometry as we approach it from the two sides V^+ and V^- if

$$g_{ij} \, dx^i \, dx^j = a_{ab}^+ \, y^a_{+,i} \, y^b_{+,j} \, d x^i \, dx^j = a_{ab}^- \, y^a_{-,i} \, y^b_{-,j} \, d x^i \, d x^j \qquad (3)$$

where $y^a_{,i} \equiv \partial y^a / \partial x^i$. Hence (3) implies

$$g_{ij} = a_{ab} \, y^a_{,i} \, y^b_{,j} \qquad (4)$$

This is known as the *matching of the first fundamental form.*

Let N^a be the unit vector normal to the bounding 3-space S given by the equations :

$$a_{ab}\, y^a{}_{,i}\, N^b = 0 \tag{5a}$$

$$a_{ab}\, N^a N^b = \pm 1 \tag{5b}$$

The positive sign in (5b) corresponds to a space-like and the negative to a time-like hyper-surface. Eqn. (5a) is equivalent to three equations. Hence it fixes the ratios of the four components N^1, N^2, N^3, N^4 but not their absolute values. Eqn. (5b) could have fixed the absolute values but as it is a quadratic equation, it has two roots corresponding to two directions of the normal. One of these corresponds to the future-directed time-line while the other to the past-directed one [Goldwirth and Katz (1995), Fayos et al (1996)]. Goldwirth and Katz has illustrated this by Fig. 1. They take a two-dimensional example of fitting a plane to a cone along the one-dimensional boundary of a circle. The pieces are numbered in the first figure. The subsequent figures 1(a)—(d) show all possible combinations of orientations of the unit normals **n** and $\mathbf{n}^{-}$.

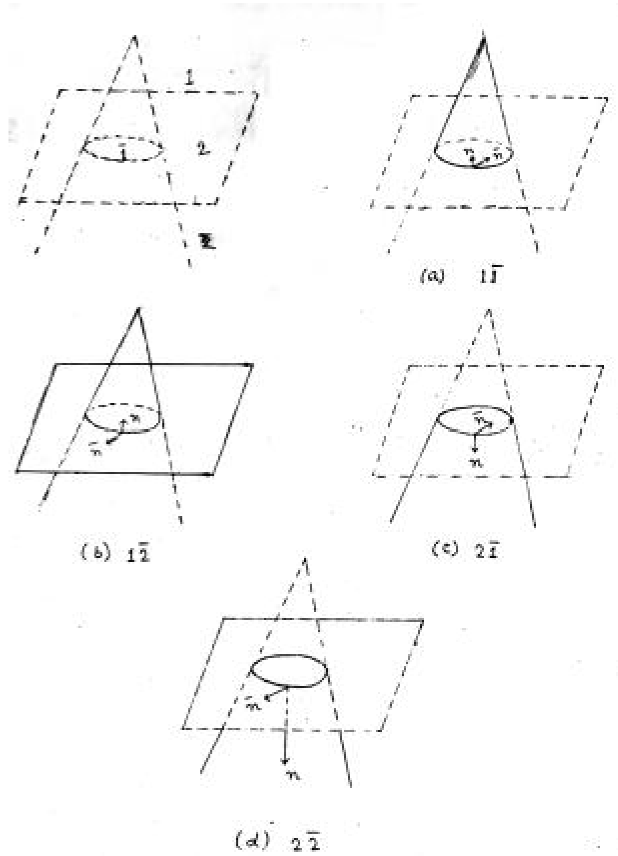


Figure 1.

Another condition to be satisfied on the boundary is that the extrinsic curvature of S relative to V^+ and V^- on the two sides should be the same. It is measured by the rate of change of the normal vector [Misner et al (1973) pp. 551-554] (see Fig. 2).

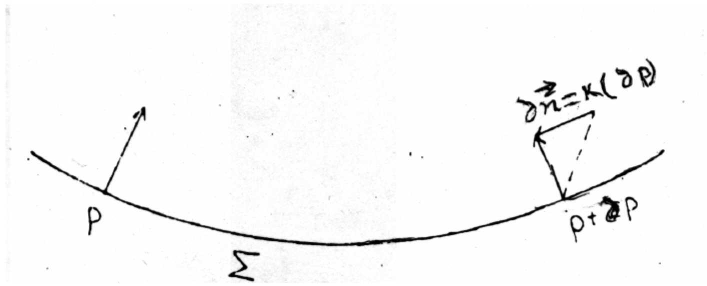


Figure 2

The extrinsic curvature is given by

$$W_{ij}^{\pm} = y^{a}{}_{;ij}\, a_{ab}{}^{\pm} N^{b}{}_{\pm} \qquad (6)$$

where

$$y^{a}{}_{;ij} = y^{a}{}_{,ij} + G^{a}{}_{bg}\, y^{b}{}_{,i}\, y^{g}{}_{,j} - G^{h}{}_{ij}\, y^{a}{}_{,h} \qquad (7)$$

Here we use a semicolon for a covariant differentiation and a comma for partial differentiation. The Christoffel symbols G with Greek indices are formed in 4-space with the metric $a_{ab}{}^{\pm}$ while those with latin indices are formed in 3-space S with the metric g_{ij} .

The invariant quantity

$$ds^{2}{}_{\pm} = W_{ij}{}^{\pm}\, dx^{i}\, dx^{j} \qquad (8)$$

is called the *second fundamental form* [Weatherburn (1957)] which should match on the two sides.

The matching conditions described above are purely geometric conditions and Einstein's field equations have nowhere been used so far.

Examples of Matching

Santos (1985) used this method to match a general spherically symmetric solution representing a shear-free collapsing non-adiabatic fluid having radial heat flow with a Vaidya metric across a 3-space S. Later Fayos et al (1992) used the above method to study the matching of the most general collapsing sphere with the Vaidya metric.[1]

Mandal and Banerji (1998) matched the Vaidya metric with the Robertson-Walker metric. We shall give this matching in a little more detail :

Let us have a Region I (4-space)with the metric :

[1] Later Fayos et al (1996) considered the general matching of two spherically symmetric space-times and the use of Penrose diagrams for the purpose.

$$ds^2_1 = a_{ab}^{+}\, y^a_{+}\, y^b_{+} = \{1 - 2m(v)/r_1\}dv^2 + 2\, dv\, dr_1 - r^2_1\,(d\, q^2 + \sin^2 q\, d\, f^2) \qquad (9)$$

This is the Vaidya metric where m is a function of the null coordinate v alone. And the coordinates are :

$$y^a_{+} = (\, r_1, q, f, v) \qquad (10)$$

In Region II we have the special Robertson-Walker metric with zero spatial curvature where it is filled with a perfect fluid with the equation of state

$$p = g\, r\,,\ \ 0 \le g \le {}^1/_3 \qquad (11)$$

$$ds_2^{\,2} = dt_2^{\,2} -\!- \ t_2^{\,2n}\,(\, dr_2^{\,2} + r_2^{\,2}\, d\, q^2\ + r_2^{\,2} \sin^2 q\ df^2\,) \qquad (12)$$

where $n = {}^2/_{\{3(g + 1)\}}$. Here the coordinates are

$$y^a_{-} = (r_2\,, q, f, t_2\,) \qquad (13)$$

We want to match the metrics (9) and (12) on the bounding 3-space S with the eqn. $r = f(t)$. The intrinsic metric will be expressed in terms of the local coordinate system :

$$x^i = (q, f, t) \qquad (14)$$

t is here the proper time. Hence the intrinsic metric of the bounding space S is

$$ds_S^{\,2} = g_{\,ij}\, d\, x^i\, d\, x^j = d\, t^2 -\!- R^2(t)\,(\, d\, q^2\ + \sin^2 q\, d\, f^2\,) \qquad (15)$$

From the matching of the first fundamental forms :

$$(d\, s_1^{\,2}\,)_S = (\, ds_2^{\,2})_S = d\, s_S^{\,2} \qquad (16)$$

we obtain

$$\dot{t}_2^{\,2} -\!- t_2^{\,2n}\, \dot{r}_2^{\,2} = 1 \qquad (17)$$

$$r_1 = r_2\ t_2^{\,n} \qquad (18)$$

$$[1 - \{2m(v)\}/r_1]\ \dot{v}^2 + 2\dot{r}_2\, \dot{v} = 1 \qquad (19)$$

Here a dot over a symbol represents its derivative with respect to t .

The unit normal vector N^a to S is given by the eqns.

$$a_{ab}\, y^a_{\,,i}\, N^b = 0 \qquad (20a)$$

$$a_{ab}\, N^a N^b = -\!-\ 1 \qquad (20b)$$

We choose the negative sign on the right of (20b) as the boundary is time-like. Solving the above eqns. We obtain two values of N^a (as explained earlier) corresponding to the two directions of the normal :

$$N_1^{\,a} = \pm\,[\, \dot{r}_1\ + (1 - \{2m(v)\}/r_1)\, \dot{v},\, 0,\, 0,\, -\!-\, \dot{v}\,] \qquad (21)$$

Using eqn.(6) we obtain

$$W_{qq} = \pm [(1 - \{2m(v)\}/r_1) r_1 \dot{v} + r_1 \dot{r}_1] = W_{ff}/\sin^2 q \tag{22}$$
$$W_{tt} = [(-m\dot{v})/r_1^2 + \ddot{v}/\dot{v}] \tag{23}$$

The unit normal vector to S in terms of coordinates in Region II is given by

$$N_2^a = \pm [t_2^{-n} \dot{t}_2, 0, 0, t_2^n \dot{r}_2] \tag{24}$$

Using eqn. (6) we obtain again

$$W_{qq} = \pm (r_2 t_2^n \dot{t}_2 + n r_2^2 \dot{r}_2 t_2^{3n-1}) = W_{ff}/\sin^2 q \tag{25}$$
$$W_{tt} = \pm (-\ddot{r}_2 \dot{t}_2 t_2^n - 2n \dot{r}_2 \dot{t}_2^2 t_2^{n-1} + n \dot{r}_2^3 t_2^{3n-1} + \dot{r}_2 t_2^n \ddot{t}_2) \tag{26}$$

Matching the second fundamental forms we obtain

$$r_2 t_2^n \dot{t}_2 + n r_2^2 \dot{r}_2 t_2^{3n-1} = \pm [(1 - \{2m(v)\}/r_1) r_1 \dot{v} + r_1 \dot{r}_1] \tag{27}$$
$$\ddot{r}_2 \dot{t}_2 t_2^n + 2n \dot{r}_2 \dot{t}_2^2 t_2^{n-1} - n \dot{r}_2^3 t_2^{3n-1} - \dot{r}_2 t_2^n \ddot{t}_2 = \pm \{(m\dot{v}/r_1^2) - (\ddot{v}/\dot{v})\} \tag{28}$$

Voids and their evolution

With the construction of bigger and bigger telescopes, astronomers have been able to get a three dimensional view of structures in the universe. Galaxies were found to form clusters like stars. The average cluster has a size ~ 5 Mpc. Further studies have revealed larger structures with sizes ~ 50 Mpc called super-clusters. In addition to these clusters and super-clusters some gaps were found which were called voids whose dimensions may be as big as 600 Mpc.[Kirshner et al (1981)]. Later evidence indicated that the voids are not completely empty but contain gas [Brosch et al (1984)] or dust [Lindley (1989)] or dark matter or radiation but are deficient in luminous matter.

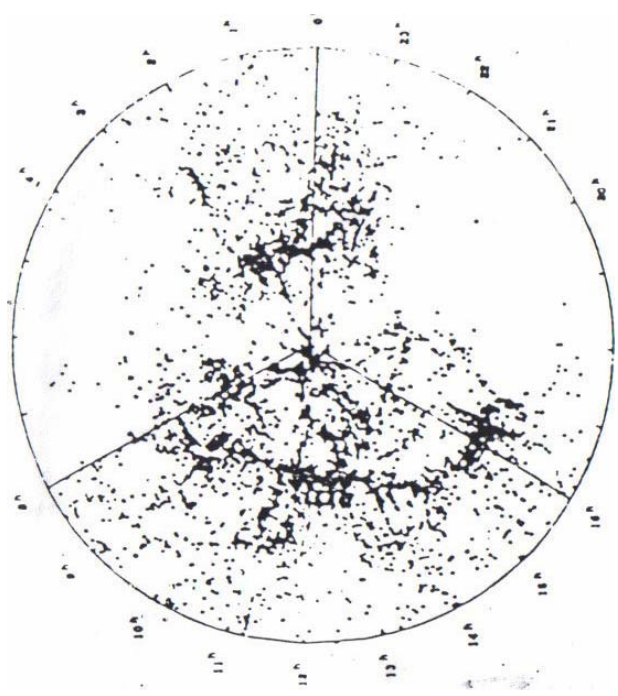


Figure 3

Mandal and Banerji (1998) considered a spherically symmetric model of the void for mathematical simplicity. The void was supposed to be formed by a central spherical region containing matter and radiation whose density is much below the average (Region I), surrounded by a spherical shell of pure radiation having the Vaidya metric (Region II). The metrics are as follows. In Region I :

$$ds_I^{\,2} = \{1 + a/(1 + x\, r_1^{\,2})\}^2\, dt_1^{\,2} \;\text{--}\; R^2\,(t_1)/(1 + x\, r_1^{\,2})^2\, \{\, dr_1^{\,2} + r_1^{\,2}\,(d\,q^2 + \sin^2 q\; d\, f^2) \qquad (29)$$

where a and x are constants. The energy momentum tensor is that of a fluid with heat flux expressed in the standard form as

$$T_m^{\;n} = (r + p)\; u_m\; u^n - p\; d_m^{\;n} \;\text{--}\; q_m u^n \;\text{--}\; u_m\; q^n \qquad (30)$$

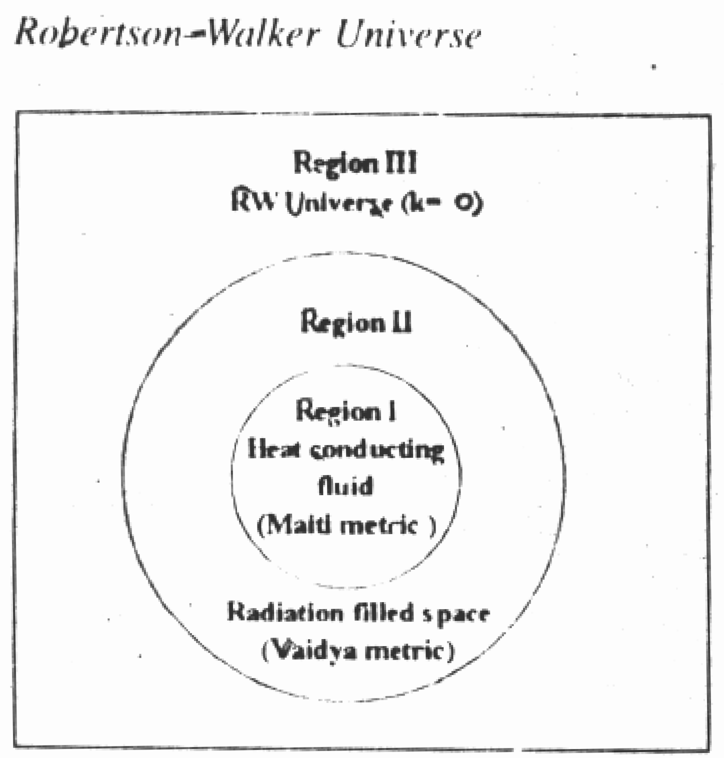
where q^m represents the heat flux vector orthogonal to the velocity vector u^m .

This is a spherical slice cut out of a special case of a solution found by Maiti(1982).

In Region II :

$$ds_{II}^{\,2} = \{\, 1 - 2m(v)/r_2\, \}d\, v^2 + 2\; d\; v\; dr_2 \;\text{--}\; r_2^{\,2}\,(d\, q^2 + \sin^2 q\; d\, f^2\,) \qquad (31)$$

This is a spherical shell cut out of the Vaidya solution. The combination of Regions I and II constitutes the void and is embedded in an FRW universe with flat space sections



Three regions of spacetime.

Figure 4.

(Region III) (see Fig. 3).

In Region III :

$$ds_{III}^2 = dt_3^2 - t_3^{2n}\{dr_3^2 + r_3^2(d\theta^2 + \sin^2\theta\, d\phi^2)\} \tag{32}$$

The space-time is filled with a perfect fluid with the equation of state $p = \gamma\rho$, $0 \leq \gamma \leq \frac{1}{3}$ and $n = {}^{2}/_{\{3(\gamma+1)\}}$. When $\gamma = 0$, $n = \frac{2}{3}$ and for $\gamma = \frac{1}{3}$, $n = \frac{1}{2}$. (32a)

From the junction conditions the above authors deduce that

$$2m = n^2 \alpha_0^3 t_3^{3n-2} \tag{33}$$

where α_0 is a constant. If $n = \frac{2}{3}$, i.e. $\gamma = 0$, m becomes a constant, hence no radiation comes from Region II. In such a case the Vaidya metric reduces to that of Schwarzschild with the transformation :

$$v = t_2 - \int dr_2/(1 - 2m/r_2) \tag{34}$$

Mandal and Banerji (1998) further deduced that t_2 of the Schwarzschild metric is related in this case to t_3 of FRW metric in Region III by the equation :

$$t_2 = \pm \int dt_3/(1 - {}^{4}/_{9}\, \alpha_0^2\, t_3^{-2/3})$$

$$= \pm [\, t_3 + {}^{4}/_{3}\, \alpha_0^2\, t_3^{1/3} + {}^{4}/_{9}\, \alpha_0^3 \ln |(3t_3^{1/3} - 2\alpha_0)/(3t_3^{1/3} + 2\alpha_0)| \tag{35}$$

If we want both t_2 and t_3 to be future-directed we must take the first sign and reject the second. The above expression agrees with that found by Dey and Banerji (1991). This choice of sign depends on the choice of direction of the normal to the bounding 3-space. In this "radiation-free" case ($p = 0$, $\gamma = 0$) the co-moving observer finds the bounding surface to be static, i.e. he finds the void to be static. In other cases the boundary m in eqn. (33) is not constant and so radiation comes out of Region II , which must be present at least near the boundary of Regions II and III. We assume that the amount of radiation coming out of Region II is small compared to the matter density in III. So after some distance the radiation is absorbed or scattered by matter so as to become non-existent. Mandal and Banerji (1998) showed that the boundary of the void satisfies the equation :

$$r_3 = u_3 = 2[\,\alpha_0 - ({}^{\gamma}/_{2})\{{}^{3(\gamma+1)}/_{(3\gamma+1)}\}\, t_3^{\{(3\gamma+1)/3(\gamma+1)\}}] \tag{36}$$

This means that the void appears to contract to a co-moving observer, a little away from the boundary in Region III.

Although Fayos et al (1991, 1992) showed that a Vaidya metric can be smoothly matched with the general FRW universe, it may appear strange as we are accustomed with the treatment of the FRW universe in co-moving coordinates where it contains a perfect fluid with no radiation. But Tupper (1981) showed that stress-energy tensors of quite different matter distributions may have precisely the same components. Suppose that $\bar{\rho}$, $\bar{p}$ be the density and pressure of the perfect fluid filling the FRW universe when we take co-moving coordinates so that the energy-momentum tensor becomes

$$T_{mn} = (\bar{r} + \bar{p})\ u_m\ u_n -- \bar{p}\ g_{mn} \tag{37}$$

On the other hand, if the fluid 's 4-velocity is v_m when the coordinates are not co-moving we may have the same components of T_{mn} as (37) with an imperfect fluid together with a null vector l_m representing radiation

$$T_{mn} = (r + p)\ v_m v_n - p\ g_{mn} - W^2 l_m l_n + P_{mn} \tag{38}$$

P_{mn} is a trace-free tensor of anisotropic pressures orthogonal to v_m . The values have been evaluated by Mandal and Banerji (1998). At least the component v^1 is non-zero near the boundary unlike u^1 and the extra term involving v^{1} is compensated for by the 3rd and 4th terms on the right of (38).

Later Ray, Chaudhury and Banerji (2000) generalized the model of the void by replacing the FRW universe with flat space sections by a general FRW space-time with non-zero spatial curvature with the metric given by

$$ds^2 = dt^2 -- S^2(t)/(1 + kr^2/4)^2 [\, dr^2 + r^2 (d\,q^2 + \sin^2 q\ d\,f^2)] \tag{39}$$

They prove that the matching conditions show that the radial coordinate of the boundary between the Vaidya and the FRW space-times is given by

$$r = u_3 = 2 \tan [\, a_0 - (g/_2)\ \sin^{-1} S^{(1 + 3g)/2}] \ \text{ for } k = +1 \tag{40a}$$

$$= 2 [\, a_0 -- (g/_2)\ \{^{3(g + 1)}/_{(3g + 1)}\} S^{(1 + 3g)/2}] \text{ for } k = 0 \tag{40b}$$

$$= 2 \tanh [\, a_0 - (g/_2)\ \sinh^{-1} S^{(1 + 3g)/2}] \tag{40c}$$

Here a_0 is a positive constant. Evidently, if $g = 0$, (i.e. the pressure vanishes), the void remains static, whatever be the value of k in the overall universe.

Discussion

In a nutshell, this model of the void would go on collapsing while the universe expands if it was created in the early universe which was not matter-dominated. This is true even if the spatial curvature of the overall universe is non-zero. But the rate of collapse depends upon the spatial curvature. The rate is fastest for k = + 1, medium for k = 0 and slowest for k = -- 1. In other words, any in-homogeneity produced in the otherwise homogeneous early universe tends to be removed. However, if a precursor of the void created in the early universe survives till the present matter—dominated epoch the collapse of the void stops. The arrow of time is assumed to point towards the future in each region.

The present day cosmologists believe that a small in-homogeneity present in the early universe at the time of decoupling of matter and radiation increased in size as the universe expands leading to the formation of structures that we see today. The present result goes against this belief unless our result is very much model-dependent. However, we have taken only one inhomogeneous region in the FRW universe. But, in actual practice, there are several under-dense as well as over-dense regions. Further, we have considered the void (or its precursor) to be spherically symmetric for mathematical convenience but a look at Fig. 3 shows that this is not at all justified. Moreover, we now know that matter (including dark matter) is only 30% of the total energy while the remaining 70% is called dark energy whose exact nature is still unknown. The dark energy is believed to be repulsive producing an accelerated universe at the present epoch. Better models of voids and filaments need to be given by theorists to explain our observations.

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