Electrodynamics

Lecture 3.3
Generalized Functions

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§ 1

Generalised functions

The standard definition of a function f is as follows. A function f is a mapping from the set $\mathcal{R}$ of real numbers (or a suitable interval of it) into the real numbers, so that it assigns a number $f(x)$, called its value, to $x \in \mathcal{R}$.

Consider the "step function" θ defined as follows :

$$\begin{aligned} \theta(x) &= 0 \quad x < 0 \\ \theta(x) &= 1 \quad x > 0 \end{aligned}$$

This is very much like an ordinary function, in fact a constant function, everywhere except the point $x = 0$ where it is not defined. The function is discontinuous at $x = 0$ and we can not define its derivative at that point.

The theory of generalised functions is a genralization of the concept of functions to include functions which may have discontinuities or singularities at some or other point of their domain of definition.

For this purpose we must look at an alternative way to define a function.

There are three different ways to define generalized functions.

1. A generalized function is defined by a sequence of ordinary functions which “tend towards” the singular function.

2. A generalized function defined indirectly when integral of its product with a smooth well behaved functions is given.

3. A generalized function is defined as boundary value of an analytic function.

All three methods are used and they complement each other.

§ 2

Sequence of functions

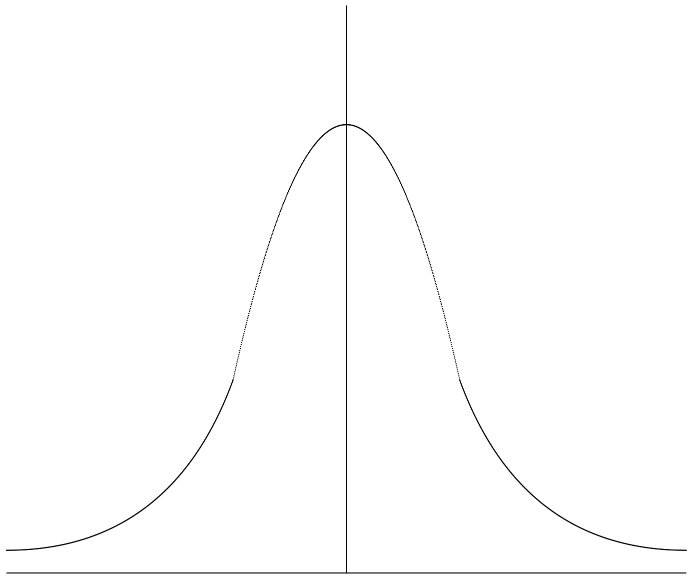
The best exampole is the Dirac delta function. The sequence of functions is chosen as

$$\delta(x) = \lim_{n\to\infty} f_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}, \qquad n = 1, 2, \ldots \tag{1}$$

By choosing $\epsilon = 1/n^2$ we can also write the above definition as

$$\delta(x) = \lim_{\epsilon\to 0} f_n(x) = \frac{1}{\sqrt{\epsilon\pi}} e^{-x^2/\epsilon}, \qquad n = 1, 2, \ldots \tag{2}$$

These functions looks like



The “area under the curve” of f_n is $\int_{-\infty}^{\infty} f_n(x)dx = 1$ (check that). And for larger values of n the functions become narrowly and sharply peaked around $x = 0$ always keeping the area under the curve equal to 1.

The Dirac delta function is the limiting function of this sequence. In the limit the function would be zero everywhere except at $x = 0$ where it would be $+\infty$.

The above sequence is not the only sequence of functions which defines the Dirac delta. There are several (in fact infinitely many) such sequences. Another example of a sequence of functions is obtained by

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$$

If you really must insist on a *sequence*, you can take $\epsilon = 1/n$

which is equivalent to $\epsilon \to 0$ as $n \to \infty$. This gives

$$\delta(x) = \lim_{n\to\infty} g_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}, \qquad n = 1, 2, \ldots$$

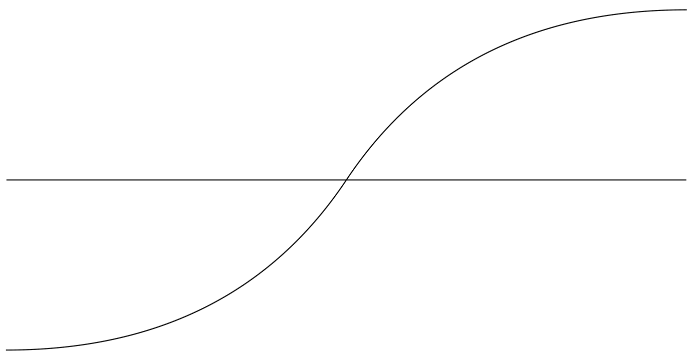
These functions also have unit area under the curve and for large values of n the functions become very sharply peaked and narrow near $x = 0$.

Actually it does not matter which particular sequence is used for the definition.

The step function can be approximated by a sequence of functions

$$\theta(x) = \lim_{n\to\infty} h_n(x) = \frac{1}{2} + \frac{1}{2}\tanh(nx)$$

The function $\tanh(nx)$ looks like



For large values of n the function becomes more and more steep at origin and for most of the positive side it is practically equal to 1 and on the negative side it is -1. Another sequence is

$$\theta(x) = \lim_{n\to\infty} k_n(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(nx) \qquad n = 1, 2, \ldots$$

where it is understood that we take the values of $\tan^{-1}(x)$ in the range $-\pi/2$ to $\pi/2$. The function $\tan^{-1}(x)$ also has graph like $\tanh(x)$, and suitable factor has been used to give the step function.

A third interesting example is the "Cauchy principal value" of $1/x$. This function is obtained by omitting the singular part of $1/x$ in a symmetrical way from the neighbourhood of $x = 0$. Let ϵ be a small number, then we define the Caucy principal value denoted by

$$P\left(\frac{1}{x}\right)$$

as the limit $\epsilon \to 0$ of the function

$$\begin{aligned} P\left(\frac{1}{x}\right) &= \frac{1}{x} \qquad (|x| > \epsilon) \\ &= 0 \qquad (|x| < \epsilon) \end{aligned}$$

Again we encounter the discontinuities. We can define the Cauchy principal value by a sequence of functions

$$P\left(\frac{1}{x}\right) = \lim_{\epsilon \to 0} \frac{x}{x^2 + \epsilon^2} = \lim_{n \to \infty} \frac{n^2 x}{1 + n^2 x^2} \qquad (\epsilon = \frac{1}{n})$$

You *must* plot these functions. The idea is that for $|x| > \epsilon$ the function behaves like $1/x$ and near $x = 0$ it is linear with a large slope (n^2). The turning point from $1/x$ to x behaviour is at $x = \epsilon = 1/n$.

We are already in a position to prove an important relation :

$$\lim_{\epsilon \to 0} \frac{1}{x \pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

The left hand side is a a complex fuction with a small imaginary part

$$\frac{1}{x \pm i\epsilon} = \frac{x}{x^2 + \epsilon^2} \mp i \frac{\epsilon}{x^2 + \epsilon^2}$$

When $\epsilon \to 0$ the first term on the right hand side becomes the Cauchy principal value, and the second term gives Dirac delta function by the definition given above. Therefore (as $\epsilon \to 0$)

$$\frac{1}{x \pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

There is another intresting fact we can derive. If a sequence f_n of functions tends to a generalized function f then we say that the sequence f'_n of derivatives of the functions defines the **derivative** f' of the generalized function. You can check that the derivative of the step function is the Dirac delta function :

$$\theta'(x) = \delta(x)$$

(Hint : use $\tan^{-1}$ definition for the step function.)

§ 3

Generalized functions defined via test functions

The method of indirect definition of a generalized function is somewhat like the way police obtains information on hard criminals through its informers who are themselves better behaved but happen to be in the company of the those wanted men.

In this method the effect of the generalized function is seen when "it is smeared with a test function". This means we integrate the generalized function to be defined with a known well behaved "test" function ϕ and give the value of the integral

$$(f, \phi) \equiv \int_{-\infty}^{+\infty} f(x)\phi(x)dx$$

The class $\mathcal{D}$ of test functions should be sufficiently large so that a knowledge of (f, ϕ) for all $\phi \in \mathcal{D}$ is enough to extract all relevant knowledge of the generalized function f.

The class of functions $\mathcal{D}$ is taken to be the set of all functions which are infinitely differentiable and which vanish outside a finite interval.

[Example and remark about the difference between analytic real and analytic complex functions.]

It is hoped that for functions with a singularity, the process of integrating with a very well behaved and smooth function ϕ will give meaningful result (f, ϕ), even though it may not be possible to define the function at all points by the usual definition.

For example, for our step function, the definition as a generalized function is

$$(\theta, \phi) = \int_0^{+\infty} \phi(x) dx$$

which is obvious in this simple case.

And the Dirac delta function is defined by

$$(\delta, \phi) = \phi(0)$$

This indirect method for generalized function can be employed to define derivatives of a generalized function which are again generalised functions.

If there was an ordinary function $f(x)$ we would write $f' = df/dx$:

$$(f', \phi) = \int_{-\infty}^{+\infty} f'(x)\phi(x) dx$$

$$
\begin{aligned}
&= [f\phi]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x)\phi'(x)dx \\
&= -(f,\phi')
\end{aligned}
$$

the first term being zero because every ϕ vanishes at $\pm\infty$.

We use the same relation to define the derivative f' of a *generalised* function f.

$$\text{Definition :} \qquad (f',\phi) = -(f,\phi')$$

As an example we can define the derivative of the step function θ as

$$(\theta',\phi) = -(\theta,\phi') = -\int_0^{+\infty} \phi'(x)dx = -[\phi(x)]_0^\infty = \phi(0)$$

Therefore θ' is a generalised function identical to Dirac delta function.

What is more δ, being a generalised function itself, has its own derivative defined

$$(\delta',\phi) = -(\delta,\phi') = -\phi'(0)$$

Thus generalised functions have derivatives of all orders defined – which is very good progress considering that it was not possible to differentiate them even once by by the usual definition.

§ 4

<u>**Fourier Transform of δ**</u>

We shall prove a very important formula, which can be written as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk$$

As things stand in this formula the integral on the right hand side is not well defined. This formula is very useful but symbolic. One way is to define it as the limit of convergent integrals

$$\begin{aligned}\delta(x) &= \lim_{\epsilon\to 0} \delta_\epsilon(x) \\ &\equiv \lim_{\epsilon\to 0} \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-\epsilon k^2}\, dk\end{aligned}$$

The integral can be done explicitly by "completing the square"

$$\begin{aligned}-\epsilon k^2 + ikx &= -\epsilon\left(k^2 - \frac{ikx}{\epsilon}\right) \\ &= -\epsilon\left[\left(k - \frac{ix}{2\epsilon}\right)^2 - \left(\frac{ix}{2\epsilon}\right)^2\right] \\ &= -\epsilon K^2 - \frac{x^2}{4\epsilon}\end{aligned}$$

where $K = k - ikx/(2\epsilon)$. The integration variable can be changed from k to K and the integral evaluated

$$\begin{aligned}\delta_\epsilon(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx-\epsilon k^2}\, dk \\ &= \frac{e^{-x^2/4\epsilon}}{2\pi}\int_{-\infty}^{+\infty} e^{-\epsilon K^2}\, dK \\ &= \frac{e^{-x^2/4\epsilon}}{2\sqrt{\pi\epsilon}}\end{aligned}$$

which gives us the delta function as $\epsilon \to 0$.