

# Introductory Cosmology

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## 1 Homogeneous and Isotropic Model

**Homogeneity:** Means that universe looks the same at each point.

**Isotropy:** Means that universe looks the same in all directions.

These are two important properties of space which are independent of each other. But isotropy at each point implies homogeneity also.

**Cosmological principle:** *Universe is homogeneous and isotropic at any given cosmic time.* The cosmological principle is supported by the observational evidence that the universe becomes smooth at large scales. The cosmological principle presents the idealized picture of the universe. The departure from homogeneity and isotropy is extremely important which led to the structure formation in the universe.

### 1.1 Hubble's law

We examine the motion of matter in a coordinate system in which it is at rest at the origin. We now ask for the velocity distribution consistent with homogeneity and isotropy.

Hubble's law:

$$\vec{v} = H(t)\vec{r} \quad (1)$$

Velocity field (1) is isotropic at  $O$ . Let us verify that (1) holds for any observer situated at a point  $A$ , The observer at  $A$  is in motion with respect to  $O$ .

$$\vec{r}' = \vec{r} - \vec{r}_A.$$

So,

$$\vec{v}' = \vec{v} - \vec{v}_A = H\vec{r} - H\vec{r}_A$$

$$\vec{v}' = H(\vec{r} - \vec{r}_A)$$

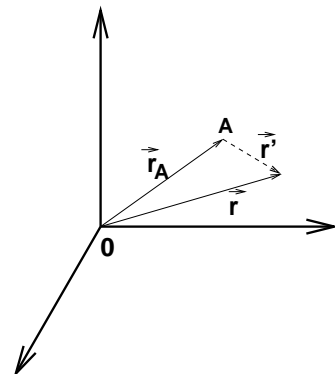
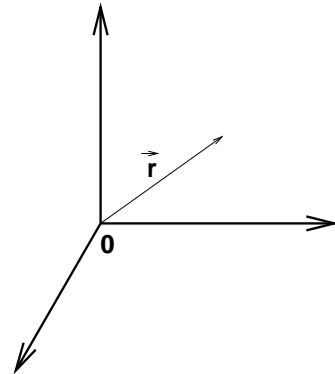
$$\vec{v}' = H\vec{r}'$$

Therefore, velocity distribution (1) is homogeneous and isotropic.

### 1.2 Local expansion

The distance between two arbitrary points changes as

$$\frac{d\vec{r}_{AB}(t)}{dt} = H(t)\vec{r}_{AB}$$



Therefore,

$$\vec{r}_{AB}(t) = \vec{r}_{AB}(t_0) \exp\left(\int_{t_0}^t H(t') dt'\right)$$

**Remark:** The dynamics will be decided by  $H(t)$ .

If  $H(t) = \text{const.}$ , then

$$\vec{r}_{AB}(t) = \vec{r}_{AB}(t_0) e^{(t-t_0)H}$$

### 1.3 Evolution of density

$$\rho = \frac{M}{\frac{4\pi}{3}R^3}$$

$$\frac{d\rho(t)}{dt} = -\frac{3M}{\left[\frac{4\pi}{3}R^3\right]} \frac{dR}{dt}$$

But  $dR/dt = v = HR$ . Therefore,

$$\frac{d\rho(t)}{dt} = -\frac{3M}{\left[\frac{4\pi}{3}R^3\right]} H = -3\rho H$$

$$\frac{d\rho}{dt} = -3\rho H$$

(2) can also be obtained from continuity equation

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v})$$

$\rho$  - function of time alone:

$$\frac{\partial \rho}{\partial t} = \rho H \vec{\nabla} \cdot \vec{r} = -3\rho H \quad (3)$$

$$\frac{\partial \rho}{\partial t} = \frac{d}{dt} \rho$$

Homogeneity and isotropy is a preserved property in time.

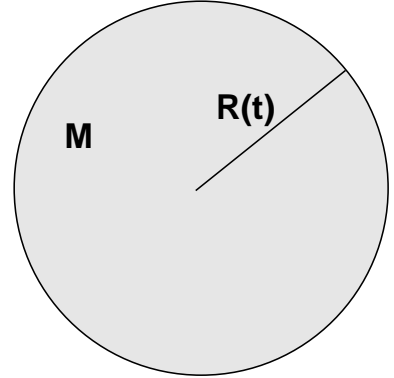
## 2 Evolution Equation

The acceleration of a particle with mass  $m$  due to the gravitational force of  $M$  is given by:

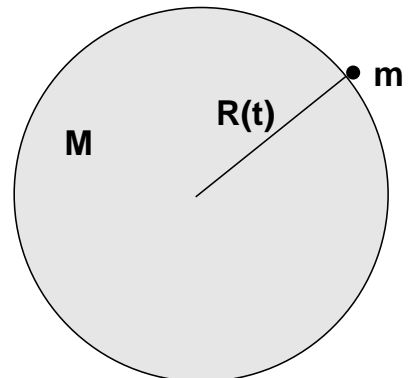
$$\frac{d^2 R(t)}{dt^2} = \frac{-GM}{R^2(t)}$$

$$\frac{d^2 R(t)}{dt^2} = \frac{d}{dt}(HR) = H \frac{dR}{dt} + R \frac{dH}{dt} = H^2 R + R \frac{dH}{dt}$$

$$\frac{dH}{dt} = -\frac{GM}{R^3} - H^2$$



(2)



$$\frac{dH}{dt} = -H^2 - \frac{4\pi}{3}G\rho \quad (4)$$

**Remark:** If  $H = \text{const}$ , eqn (4) is inconsistent. Infact  $H = \text{const}$  (with  $\frac{dh}{dt} = \frac{\dot{R}}{R} - H^2$ ) would imply  $\ddot{R}(t) > 0$ , which cannot come from eqn (4).

**Friedman Equation:** Multiply eqn (3) by  $dR(t)/dt$ :

$$\begin{aligned} \frac{dR}{dt} \left( \frac{d^2 R}{dt^2} \right) &= -\frac{GM}{R^2} \frac{dR}{dt} \\ \frac{1}{2} \frac{d}{dt} \left( \frac{dR}{dt} \right)^2 &= \frac{d}{dt} \left( \frac{GM}{R} \right) \\ \frac{d}{dt} \left[ \frac{1}{2} \left( \frac{dR}{dt} \right)^2 - \frac{GM}{R} \right] &= 0 \\ \boxed{\frac{1}{2} \left( \frac{dR}{dt} \right)^2 - \frac{GM}{R} = A = \text{const}} & \quad (5) \\ M &= \frac{4}{3}\pi\rho R^3 \end{aligned}$$

**Determining the constant  $A$**

$\{t_0, \rho_0\}$  present epoch and select a value of  $R = R_0$  at  $t = t_0$  for the sphere.

$$\begin{aligned} A &= \frac{1}{2} \left( \frac{dR}{dt} \right)^2_{t=t_0} - \frac{4}{3}\pi G R_0^2 \rho_0 \\ &= \frac{1}{2} H_0^2 R_0^2 - \frac{4}{3}\pi G R_0^2 \rho_0 \end{aligned}$$

$$\boxed{\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3} \rho R^2 - \frac{8\pi G}{3} R_0^2 \left[ \rho_0 - \frac{3H_0^2}{8\pi G} \right]} \quad (6)$$

This is called Freidman equation. From continuity equation

$$\boxed{\rho(t) = \frac{\rho_0 R_0^3}{R^3}}$$

## 2.1 General character of the solution of (6)

At present,  $\frac{dR}{dt} > 0$  implies that  $R$  was smaller in the past, but  $\frac{8\pi G}{3}\rho R^2$  was larger, so  $\frac{dR}{dt}$  was larger in the past:

$$\boxed{R(t_0) = 0, \quad \left. \frac{dR}{dt} \right|_{t=t_0} = +\infty \quad t = t_0}$$

Explanation

$$R \geq 0 \text{ by definition, } \ddot{R}(t) < 0, \rho > 0$$

Hence  $R(t)$  was smaller and smaller as we go into past deeper and deeper, and  $dR/dt$  becomes larger and larger. Consequently, there was an epoch, say  $t = 0$ , when

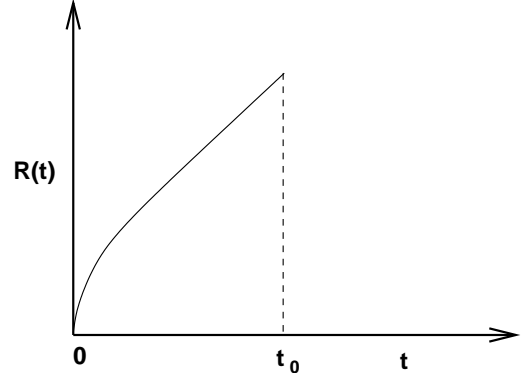
$$R(t=0) = R(0) = 0 \quad \left. \frac{dR}{dt} \right|_{t=0} = \infty$$

**Critical density:** The prediction of the future depends upon the sign of  $[\rho_0 - 3H_0^2/8\pi G]$  or how the present density compares with the critical density  $\rho_c$ :

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

We also defined the dimensionless density parameter  $\Omega_0$ :

$$\Omega_0 \equiv \frac{\rho_0}{\rho_c} = \frac{8\pi G \rho_0}{3H_0^2}$$



## 2.2 Classification of the solution

I  $\rho_0 > \rho_c$  The second term in eqn (6) is positive. As  $R$  increases, the first term decreases and eventually becomes equal to the second term at a particular time. The RHS of eqn (6) then vanishes and expansion ceases, and contraction begins.

II  $\rho_0 < \rho_c$  RHS of equation (6) is positive, leading to expansion forever.  
As  $t \rightarrow \infty, R \rightarrow \infty$

$$\left. \frac{dR}{dt} \right|_{t=\infty} = \left[ \frac{8\pi G}{3} R_0^2 (\rho_c - \rho) \right]^{1/2}$$

III  $\rho_0 = \rho_c$  Expansion continues without bound.

## 2.3 Maximum Age Estimate

$t = 0, R = 0$ . Suppose the expansion rate is constant and given by the present value of Hubble parameter,

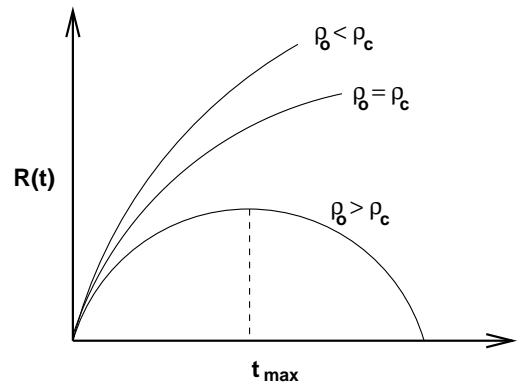
$$R(t_0) \equiv R_0 = \left( \frac{dR}{dt} \right)_{t=0} t_0 = H_0 R_0 t_0$$

$$t_0 = \frac{1}{H_0}$$

$$T_0 \simeq h^{-1} 9.8 \times 10^9 \text{ years}$$

$$H_0 \simeq 100 \text{ km s}^{-1} \text{ mpc}^{-1} \quad h \simeq 1 - 0.5 \quad t \simeq 9.8 \times 10^9 h^{-1} \text{ years}$$

$$0.37 < H_0 t_0 < 1.47$$



### 3 Solutions of Evolution Equation

$$\rho_0 = \rho_c \text{ or } \Omega_0 = 1$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 \quad \boxed{\rho(t) = \frac{\rho_0}{R^3}R_0^3}$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0 R_0^3}{R} \quad dRR^{1/2} = \text{const}.dt$$

$$R(t) = R(t=0)(t/t_0)^{2/3} \Rightarrow \boxed{R_0 \left(\frac{t}{t_0}\right)^{2/3} = R(t)}$$

$$R(t) \propto t^{2/3}$$

$$\dot{R}(t) \propto \frac{2}{3}t^{-1/3} \Rightarrow \frac{\dot{R}(t)}{R} = \frac{2}{3t}$$

Therefore,  $t_0 = \frac{2}{3H_0}$ .

$$\rho(t) = \frac{\rho_0 R_0^3}{R^3(t)} = \frac{\rho_0 R_0^3}{R_0^3(t/t_0)^2} = \frac{3H_0^2}{8\pi G t^2} t_0^2 = \frac{3H_0^2}{8\pi G t^2} \frac{4}{9} \frac{1}{H_0^2} = \frac{1}{6\pi G t^2}$$

$$\boxed{\rho(t) = \frac{1}{6\pi G t^2}}$$

#### 3.1 Pressure corrections: Relativistic effects

$$\frac{\partial \rho}{\partial t} + 3H \left( \rho + \frac{P}{c^2} \right) = 0 \quad (7)$$

$$\frac{\ddot{R}}{R} = \frac{-4\pi G}{3} \left( \rho + \frac{P}{c^2} \right) \quad (8)$$

$$\frac{\dot{R}^2}{2} - \frac{4\pi G}{3} R^2 \rho = A \quad (9)$$

Consistency check: Differentiating (9), we get

$$\dot{R}\ddot{R} - \frac{4\pi G}{3} R^2 \dot{\rho} - \frac{4\pi G}{3} \times 2 \times \rho \dot{R}R = 0$$

$$\dot{R} = HR$$

$$\dot{R}\ddot{R} + \frac{4\pi G}{3} R^2 H \left( \rho + \frac{P}{c^2} \right) - \frac{8\pi G}{3} \rho \dot{R}R = 0$$

$$\frac{\ddot{R}}{R} + \frac{4\pi G}{3} \left( \rho + \frac{P}{c^2} \right) - \frac{8\pi G}{3} \rho = 0$$

$$\boxed{\frac{\ddot{R}}{R} = \frac{-4\pi G}{3} \left( \rho + \frac{P}{c^2} \right)}$$

System (7), (8) and (9) are consistent.

### 3.2 Solution of Equations in Case of Radiation Domination

**Equation of State:** Out of the eqs (7), (8) and (9), only two are independent. Eq. (9) can be obtained from (7) and (8). These eqs can be solved and  $\rho(t)$ ,  $R(t)$  can be uniquely determined provided the *equation of state* (= relation between  $\rho$  and  $P$ ) is given. This relation, in simple cases, can be written as

$$P = \omega \rho c^2, \quad \omega = \begin{cases} 0 & \text{Dust} \\ \frac{1}{3} & \text{Radiation} \\ -1 & \text{Cosmological constant} \end{cases}$$

$$\omega = \frac{1}{3}, \quad \rho_0 = \rho_c$$

$$\frac{\partial \rho}{\partial t} + 3H \left( \rho + \frac{P}{c^2} \right) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + 4H\rho = 0$$

$$\boxed{\rho(t) = \rho_0 \frac{R_0^4}{R^4(t)}}$$

$$\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho \quad \Rightarrow \quad \dot{R}(t) \propto \frac{1}{R(t)}$$

$$\boxed{R(t) = R_0 \left( \frac{t}{t_0} \right)^{1/2}}$$

$$H(t) = \frac{1}{2t} \quad \Rightarrow \quad \boxed{t_0 = \frac{1}{2H_0}}$$

$$\rho(t) = \frac{3}{32\pi G} \frac{1}{t^2}$$

## 4 Standard Form of Evolution Equations

**Comoving Coordinates:** These coordinates are carried along with the expansion. Since the expansion is uniform, the relation between the Physical and Comoving Coordinates is given by

$$\vec{r}(t) = a(t)\vec{x} \quad a(t) \rightarrow \text{scale factor}$$

The uniformity of expansion is encoded in the scale factor.

$$\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3} \rho R^2 - \frac{8\pi G}{3} R_0^2 [\rho_0 - \rho_c]$$

Put  $R(t) = a(t)x$

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{H}{x^2 a^2}$$

Let  $Kc^2 = \frac{A}{x^2}$  or  $K = \frac{A}{x^2c^2}$ . Dimensionally

$$[K] = \frac{L^2}{T^2} \frac{1}{L^2 \frac{L^2}{T^2}} = L^{-2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2}$$

Also, from (8)

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3} \left(\rho + \frac{3P}{c^2}\right)$$

Putting  $c = 1$

$$\boxed{\frac{\partial \rho}{\partial t} + 3H(\rho + P) = 0}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$$

$$\boxed{\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3P)}$$

## 4.1 Thermal History

For  $K = 0$ ,  $\rho_r(t) = \frac{3}{32\pi G t^2}$

$$\rho_r(t) = \alpha T^4$$

$$\boxed{T = \left(\frac{3}{32\pi G \alpha}\right)^{1/4} t^{-1/2}}$$

$$\boxed{T_K = 1.5 \times 10^{10} t_{sec}^{-1/2}}$$

$$\rho_r(t) \propto \frac{1}{a^4} \quad \Rightarrow \quad T(a) = T_0 \left(\frac{a_0}{a}\right)$$

## 4.2 Decoupling

Decoupling takes place at a temperature equal to the binding energy of Hydrogen atom.

$$3k_B T_d = 13.6 eV$$

$$T_d = \frac{13.6 eV}{3k_B} \approx 5 \times 10^4 K \quad (1eV \simeq 10^4 K)$$

However, temperature in reality is much smaller than this

$$T_d \simeq 3000 K \quad \Rightarrow \quad \frac{a_0}{a_d} = \frac{3000}{T_0} \simeq 1000$$

Since  $T_K = 1.5 \times 10^{10} T^{-1/2}$ , decoupling time

$$t_d = 10^{13} \text{ sec} \simeq 3 \times 10^5 \text{ yrs.}$$

**Remark:**

$$\begin{aligned} \frac{\Omega_r}{\Omega_m} &\propto \frac{1}{a} \\ &= \frac{\Omega_r^0}{\Omega_m^0} \frac{1}{a} \approx \frac{4 \times 10^{-5}}{\Omega_m^0} \frac{1}{a} \end{aligned}$$

$$\begin{aligned} a_{eq} &\simeq 2.4 \times 10^4 \Omega_0 \\ t_{eq} &\simeq 2.5 \times 10^3 \Omega_0^{-3/2} \text{ years} \end{aligned}$$

## 5 Cosmological Constant

Einstein introduced the *cosmological constant* to make the universe static (later described by him as his “biggest blunder”).

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} \rho - \frac{K}{a^2} + \frac{\Lambda}{3} \\ \rho_\Lambda &= \frac{\Lambda}{8\pi G} \quad P_\Lambda = \frac{-\Lambda}{8\pi G} \end{aligned}$$

$\Lambda > 0$  and  $\rho$ :  $H = 0$

$$a(t) \propto e^{\frac{\Lambda}{3}t}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} (\rho_\Lambda + 3P_\Lambda) = +\frac{8\pi G}{3} \rho_\Lambda$$

$\ddot{a} > 0 \Rightarrow$  accelerated expansion. **Inflation**

$$\begin{aligned} H^2 &= \frac{8\pi G}{3} \rho - \frac{K}{a^2} + \frac{\Lambda}{3} \\ 1 &= \Omega_m + \Omega_\Lambda - \frac{K}{a^2 H^2} \end{aligned}$$

$$\boxed{\Omega_m + \Omega_\Lambda - 1 = \frac{K}{a^2 H^2}}$$