Quantum Hall Effect

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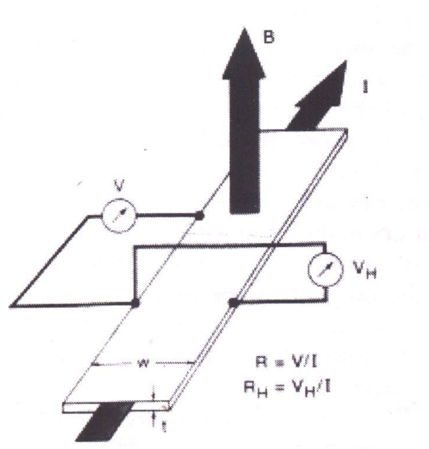
15th June, 2004

Introduction

The quantum hall effect is one of the most striking and surprising developments to occur in physics in the last 20 years. The integer quantum hall effect manifests itself as a series of plateaus in hall resistance, R_H of materials containing 2-D electron systems. R_H is precisely given by-

$$R_H = h/je^2 = (25812.81/j)\Omega \tag{1}$$

We imagine a conductor through which current I is flowing. If it is not a superconductor, there is a voltage drop V along the direction of the current flow. But if we try to measure the voltage perpendicular to the current flow, no such voltage will be observed. If a magnetic field B is applied perpendicular to the current flow, a voltage V_H will be produced perpendicular to the current flow and the corresponding resistance is given by-



$$R_H = V_H/I \tag{2}$$

This 'two dimensional' actually means three dimensional but is very thin of the order of around 100 A. The most common example of such a system is MOSFET and semiconductor hetrojunction The important thing is that electrons can move freely in a 2-D plane but not in a perpendicular direction.

There are two fundamental ingredients for the understanding of the hall effect:

1. Landau quantization of states induced by the magnetic field on the 2-D electron motion. 2. Localization

If we plot a graph between B and R_H, it should come out to be a straight line (classically)

But if we have very thin conductor and at very low temperatures, at around T=50 mK, it exhibits a series of plateaus.

Classically, 2D Classical Hall Formula is given by-

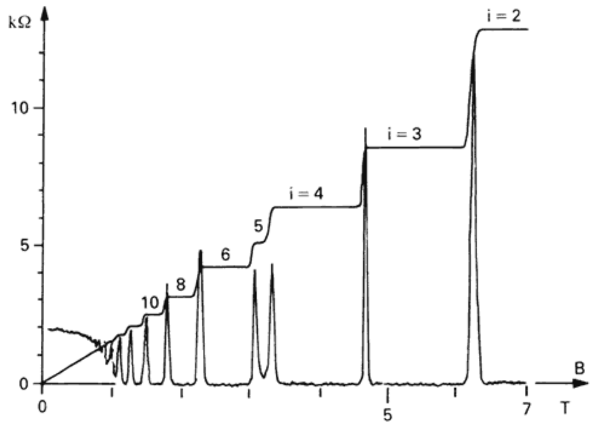
$$R_H = \frac{B}{N_Sec} \tag{3}$$

where N_S is the concentration of charge carriers (number of carriers per unit area). One will notice that the sign of R_H depends on the sign of e, the charge of carriers. So, Hall effect also gives information on whether the carriers are electrons or holes.

Experimentally, we see that hall resistance is quantized at very low temperatures and at the plateau region, it is given by

$$R_H = h/fe^2 \tag{4}$$

where f = 1, 2, 3... This was discovered by Von Klitzing and he got Nobel Prize for this discovery in 1985.



2-D electrons in a magnetic field

Origin of Landau Levels

Let the electrons be confined to the x-y plane and the magnetic field be parallel to the z-axis

The Hamiltonian for such a system is given by-

$$H = \frac{(\mathbf{p} + ie\mathbf{A}/c)^2}{2m} = \frac{[(p_x + eA_x/c) + (p_y + eA_y/c)]}{2m} \tag{5}$$

where **A** is the vector potential associated with the magnetic field **B** i.e., $\mathbf{B} = \nabla \times \mathbf{A}$. Under gauge choice, we are free to choose $\mathbf{A}$ under which $\mathbf{B}$ remains unchanged. If we choose $\mathbf{A}$ = -yB$\hat{\imath}$ and substitute into equation (5), we get-

$$H = \frac{p_y^2}{2m} + \frac{(p_x - eyB/c)^2}{2m} \tag{6}$$

Since the coordinate x doesnot appear in H and since we know the commutators [y,p_x] = [p_y,p_x] = 0, it follows that [H,p_x] = 0. This means that p_x is a good quantum number and may be regarded as a constant parameter. We define k=$p_x/\hbar$. The eigen functions of the problem must now smultaneously satisfy:

$$p_x\psi(x, y) = \hbar k\psi(x, y) \tag{7}$$

$$H\psi(x, y) = E\psi(x, y) \tag{8}$$

Since the operator p_x is given by -i$\hbar\partial/\partial$x, the first equation yields the desired result:

$$\psi(x, y) = e^{ikx}\phi(y) \tag{9}$$

The Quantum Hall Effect has analogy to the one-dimensional simple harmonic oscillator. We can see that the Schrodinger equation Hψ = Eψ reduces to the one dimensional problem:

Equation (8) is the schrodinger equation and its solutions determine the energy spectrum. We have

$$\begin{aligned} H\psi &= \frac{[(p_x - eyB/c)^2 + {p_y}^2]e^{ikx}\phi(y)}{2m} \\ &= \frac{e^{ikx}[(\hbar k - eyB/c)^2 + {p_y}^2]}{2m} \\ &= e^{ikx}\left[\frac{e^2B^2}{2mc^2}(y - \hbar kc/eB) + \frac{p_y^2}{2m}\right]\phi(y) \\ &= e^{ikx}\left[\frac{m\omega_c^2}{2}(y - y_0)^2 + \frac{p_y^2}{2m}\right]\phi(y) \end{aligned} \tag{10}$$

where $\omega_c = \frac{eB}{mc}$ and $y_0 = \hbar kc/eB$. Here, ω_c is known as the cyclotron frequency. The right hand side must equal $E\psi = Ee^{ikx}\phi(y)$. thus the differential equation is obtained(after replacingp_y by -i$\hbar\partial/\partial y$):

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega_c^2}{2}(y-y_0)^2\right]\phi(y) = E\phi(y) \tag{11}$$

We begin by recalling the ordinary one dimensional Schrodinger equation for a simple harmonic oscillator with frequency ω:

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega_c^2}{2}z^2\right]\phi(z) = E\phi(z) \tag{12}$$

where z is a dummy coordinate. Our problem looks very similar except we have a term $(y-y_0)^2$ instead of just y^2. But this simply means that the center of the oscillations is at $y = y_0$ rather than at $y = 0$. As we can see by making the change of the variable $z = y - y_0$. For any given y_0, i.e., for every k our problem is exactly like that the corresponding 1D simple harmonic oscillator. Most important, the energy levels for the two problems must be the same. From our knowledge of simple harmonic oscillators, it should be no surprise that

$$E_n = (n + 1/2)\hbar\omega_c \tag{13}$$

with n = 0,1,2...

This is a remarkable result. The allowed energy levels for a free 2D electron moving in a magnetic field are identical to a fictitious 1D simple harmonic oscillator. Note these levels are discrete; with the magnetic field the electron's energy is a continuous variable.

Note also that the energy levels do not depend on the value of y_0 or therefore k. The "momentum" k in x direction creates no kinetic energy! Each value of n can have, apparently, any value of k and thus the energy levels are highly degenerate.

Thus the magnetic field has induced a great condensation of the continuous energy spectrum of a free particle in 2D into a discrete set of highly degenerate levels. these levels are known as Landau levels, are equally spaced by the cyclotron energy, $\hbar\omega_c$, which is itself proportional to the magnetic field strength. The gaps between the levels are void of electronic states. For 3D electrons, no such gaps occur.

Estimating Degeneracy

We have :

$$y_0 = \frac{\hbar kc}{eB}.$$

Along the x-axis, the electron behaves like a particle in a box, and its k values are given by $k = \frac{2\pi n}{L_x}$, with n=0,1,2.....

$$y_0 = \frac{\hbar 2\pi nc}{L_x eB}$$

Let $y_0 max$ be the maximum value y_0 can take, and correspondingly, N is the maximum value of n.

$$y_{0max} = \frac{\hbar 2\pi Nc}{L_x eB}$$

So, N is the number states available in each Landau level. But the maximum value of y_0 is L_y, the physical dimension of sample (so that the electron stays within the sample):

$$L_x L_y = \frac{\hbar 2\pi N c}{eB}$$

$$\frac{N}{L_x L_y} = \frac{eB}{\hbar 2\pi c} = \frac{eB}{hc}$$

But $N/L_x L_y$ is the number of states per unit area, in a Landau level. Let us call that $N_0 =$ degeneracy per unit area.

$$N_0 = \frac{eB}{hc} = \frac{B}{\phi_0}$$

where $\phi_0 = hc/e$

If exactly j Landau levels are completely filled, the total number of of electrons per unit area should be $j\times$degeneracy per unit area:

$$N_s = \frac{feB}{\hbar}$$

Now, the Hall resistance is given by

$$\begin{aligned} R_H &= \frac{B}{N_S ec} \\ &= \frac{Bhc}{feBec} \\ &= \frac{h}{fe^2} \end{aligned} \tag{14}$$

Impurities play a particularly important role in the ordinary electrical resistance of metals and semiconductors. Much of the energy dissipation that characterizes resistance occurs when the electrons are scattered by collisions with impurity atoms or defects in the crystal lattice, Paradoxically, the presence of impurities is what leads to the disappearance of electrical resistance and the plateaus in Hall resistance that constitutes the quantized Hall effect.

In the presence of impurities, the many independent quantum states hat make up a given Landau level are no longer precisely equal in energy. In a semi-classical explanation, one might say, for example, that in certain of the quantum states, the electron is more likely to be found near an impurity atom that has an excess of positive charge. Such states would be slightly stabler than others in the same Landau level and would have a slightly lower energy. The single energy level that makes up a given Landau level in a pure crystal is thus spread out, in the presence of impurities, into a band made up of many distinct energy levels.

The various quantum states in each energy band can be divided into three general classes. The states near the bottom of each band, that is those of lowest energy are each localized in small region of sample. Near the top of each band, are the high energy localized states perhaps in the region around impurity atoms that have acquired electrons and have excess of negative charge.

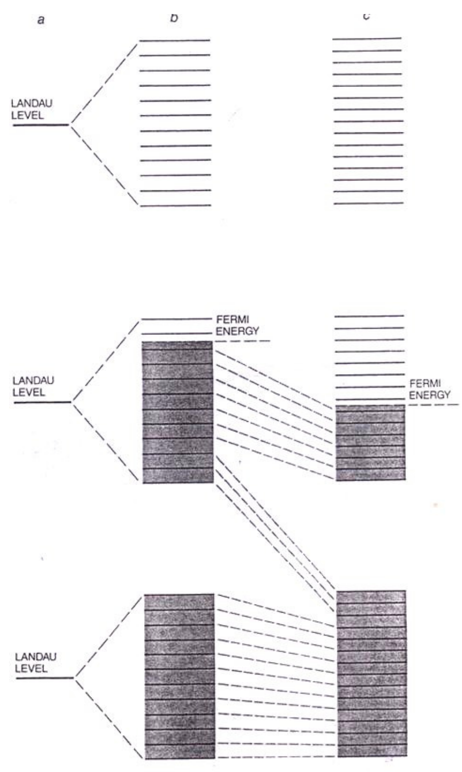
Near the center of each energy band, are the extended states which are spread out over a large region of space.

The localized and extended states are distinct because the electrons in the localized states cannot move very far and cannot carry current.

According to the Pauli exclusion principle, one of the foundations of quantum mechanics, no two electrons can occupy the same quantum-mechanical state. When the system is at its lowest possible energy, then each of the quantum states contain exactly one electron and each state above that level contains no electrons The energy of the highest filled electron is known as the Fermi energy.

When there is a voltage difference between two edges of the sample, it is not actually possible to define a single Fermi energy for the entire plane of conduction electrons; the Fermi energies vary from point to point. In each region of space, the states with energies below the local Fermi level are occupied and above it are empty.

Suppose a current is flowing along a sample in a perpendicular magnetic field and the Fermi level of the sample's electrons is in the sub-band of localized states near the top of some Landau band. In this case all the extended states and the low energy localized states of the Landau band will be completely filled and some of the high energy localized states will be occupied as well. Now suppose that the strength of the magnetic field is gradually increased and that at the same time, the current is continuously adjusted in such a way that the Hall voltage between the sample's two edges remains constant.



Many of the newly available states will be below the local Fermi level and so electrons from higher energy occupied states will drop down to fill them; these electrons will in general come from the high energy localized states that are near the Fermi level. As these states are vacated the Fermi level-the energy of the highest occupied level-descends to a lower position within the Landau band. As long as the Fermi level remains in the sub-band of high-energy localized states, all the extended states within the Landau band remain fully occupied.

Because the number of independent quantum states per unit area is directly proportional to the applied magnetic field, as the magnetic field increases, the number of independent quantum states per unit area is directly proportional to the applied magnetic field, as the magnetic field increases, the number of independent quantum states in each Landau level increases proportionately:

$$N_0 = eB/hc \tag{15}$$

in every region of space, within the sample, additional quantum states that have roughly the same energy as neighboring states become available.

Because an electron trapped in a localized state cannot move through the sample,the changing fraction of localized states that are filled has no effect on the sample's large scale electrical properties. The amount of current flowing in the sample therefore remains constant as long as the sule-band of extended states is completely filled : although the increased magnetic field slows the forward motion of any current carrying electrons, this effect is precisely cancelled by the increase, due to the newly created extended states, in the number of electrons available to carry current.

Since the Hall voltage is being held constant,the fact that the current does not change as the magnetic field is varied implies that the Hall resistance also remains constant.

Whenever the Fermi level is in the sub-band of localized states, then, the Hall Resistance remains the same even when the magnetic field is varied. This is the plateau in Hall resistance that is characteristic of the quantized Hall effect. Eventually, as the strength of the magnetic field is increased, the supply of electrons in the high-energy localized states will be exhausted and the Fermi level must drop into the sub-band of extended states. As the Fermi level descends through the sub-band of extended states, some of them are vacated. Because the current carrying sub-band is then only partially occupied, the amount of the current flowing decreases and the Hall resistance therefore increases. The Hall resistance continues to increase when the magnetic field is increased, as long as the Fermi level remains in the sub-band of extended states.

If the magnetic field is increased further, eventually the extended states within the Landau band will all be emptied and the Fermi level will once again enter a sub-band of localised states : the low-energy localized states at the bottom of the landau band. If there is at least one full landau band below the Fermi level, the extended staes in that band will be able to carry a current and the Quantized Hall effect will once more be observed. Because the extended states of one landau band have benn comletely emptied, however, the number of sub-bands of occupied states has been reduced by one.

In this model, it is easy to understand why the ratio of Hall resistances at any two plateau should equal a ratio of integers. The reason is that for any given Hall voltage. the current is directly proportional to the number of occupied sub-bands of extended states and on each plateau, an integral nuumber of such sub-bands is filled.

If the magnetic field is increased still further, the Fermi level will move down through the regions of localized states at the bottom of one Landau band and the high-energy localized states at the bottom of one Landau band and into the high-energy localized states at the top of the next Landau band. The Hall resistance will remain constant at its new plateau value until the Fermi level reaches the region of extended states in the middle of this next Landau band.

Laughlin Wavefunction

An earlier attempt to explain the FQHE, the so-called quasi-particle hierarchy(QPH) approach, started with the work af Laughlin, in which he proposed an ansatz wave function to describe the correlated electron liquid at $\nu = 1/(2m+1) = 1/3, 1/5, 1/7, ...$, where m is an integer. It was compared by Laughlin and others with the exact numerical ground-state wave function of few electron systems and was found to be extremely accurate. Laughlin also constructed wave functions for the quasi-particle excitations and made compelling arguments that there was a finite gap, resulting in FQHE with $f = 1/(2m+1)$. The Laughlin wavfunction has the form:

$$\phi_{1/(2p+1)} = \prod_{j<k} (z_j - z_k)^{2p+1} \exp\left[-\frac{1}{4}\sum_l |z_l|^2\right]$$

To explain the other fractions, Haldane and Halperin proposed iterative hierarchical schemes, which conjectured that "daughter" states occur when the quasi-particles of a "parent" state themselves form a Lauhlin-like state. For example, 1/3 produced daughters at 2/5 and 2/7, which in turn generated 5/17, 3/11, 5/13, and 3/7, and so on. In this step-by-step manner, the QPH scheme allows for FQHE at all odd-denominator fractions starting from $f = 1/(2m+1)$.

The QPH approach was somewhat speculative and not entirely satisfactory. The fact that a good description was available for $f = 1/(2m+1)$ but not for other fractions was puzzling; given the qualitative similarity of the observations of various fractions , one would have thought that once the origin of the FQHE was resolved, it should explain all fractions on a more or less equal footing. This indicated that the physics of the Laughlin wave function was itself not fully understood. There were several attempts to elucidate the relevant correlations in the Laughlin wave function. Girvin and MacDonald related it, by a singular gauge transformation, to a boson wave function, which possessed algebraic off-diagonal long-range order. Zhang, Hansson, and Kivelson and Read proposed a mean field theory in which the Laughlin wave function was viewed as a Bose condensate. These theories, however, also did not shed any new light on the other fractions.

Composite Fermion Theory

One of the crucial steps of the composite fermion theory was to allow the use of higher LLs even in the discussion of FQHE. The IQHE can be understood in term of the non-interacting electron. It is ultimately a consequence of the quantisation of the single electron energy into the Landau levels, which produces a non-degenerate amny particle ground state when an integer number of LLs are occupied.

The FQHE cannot be explained in terms of non-interacting electrons since the ground state of non-interacting electrons at a fractional filling is highly degenerate.

One might expect a liquid of interacting electrons to behave in a very complex manner. It often resembles a weakly interacting gas of particles different from electrons which may be called the quasi particles of the system.

In our case, the strongly correlated liquid of electrons is equivalent to a weakly interacting gas of particles called composite fermion.

A composite fermion is an electron carrying an even number of vertex is defined so that an electron aquires a phase of 2π upon traversing a closed loop around it.

The basic hypothesis of the composite fermion theory is that the electrons in the lowest landau level avoid other most efficiently by capturing an even number of vortices of the wave function and transforming into composite fermions.

Composite fermions move in an effective magnetic field, because the phases generated by the vortices partly cancel the Aharonov Bohen Phases originating from the external magnetic field.

Degeneracy of each Landau level per unit area is given by B/ϕ_0 where $\phi_0 = hc/e$, quantum of magnetic flux. This implies that the number of occupied Landau levels, called filling factor is given by

$$\nu = \rho\phi_0/B,$$

where $\rho =$ density of electron. The strongly interacting electron in a strong magnetic field transform into weakly interacting composite fermions in a weaker magnetic field given by

$$B^* = B - 2p\rho\phi_0 \tag{16}$$

We have

$$\nu = \rho\phi_0/B$$

Therefore

$$\nu^* = \nu\phi_0/B^*$$

This gives

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}$$

We start by considering interacting electron in a traverse magnetic field B. Now attach to each electron an infinitely thin, massless magnetic solenoid carrying 2p flux quanta pointing antiparallel to B, turning into a composite fermion. The additional magnetic field is given by $-2p\rho\phi_0$ giving $B^* = B - 2p\rho\phi_0$

An electron making a clockwise closed loop around a magnetic solenoid carrying field B acquires a "Aharonov-Bohm" phase equal to $2\pi BA/\phi_0$ where A = area of the loop

For composite fermion, it is

$$2\pi B^* A/\phi_0$$

How do vortices cancel part of external B?

Consider a path in which one particle executes a counter clockwise loop enclosing an area A, with all other particles held fixed.

Also we have the phase $-2\pi 2p\rho A$ coming from $2p\rho A$ encircled vortices. Now equating the Aharonov-Bohm phase $2\pi BA/\phi_0$ and the phase $-2\pi 2p\rho A$ to an effective Aharonov-Bohm phase $2\pi B^* A/\phi_0$, we have

$$\frac{2\pi B^* A}{\phi_0} = \frac{2\pi BA}{\phi_0} - 2\pi 2p\rho A \tag{17}$$

$B^* = B - 2p\rho\phi_0$

The wave function of an non interacting composite fermions at $\nu^*, {\phi_{\nu^*}}^{CF}$ is constructed simply by taking the known wave functions of non interacting electrons at ν^*, ϕ_{ν^*} and attaching 2p vortices to each electron.

$${\phi_{\nu^*}}^{CF} = \prod_{j<k}(z_j - z_k)^{2p}\phi_{\nu^*} \tag{18}$$

where $z_j = x_j + iy_j$ denotes the position of the jth particle and multiplication by Jastrow factor attaches 2p vortices to each electron and convert it into composite fermions.

Basic Concepts of the Composite Fermion Theory

The central achievement of the composite fermion theory is to identify the true particles of the Landau level liquid. These are called composite fermions. These are electrons carrying an even number(2p) of vortices of the many body wave function.

1. Electrons capture 2p vortices to become composite fermions.

2. Composite fermions are weakly interacting.

3. The most remarkable outcome of formation of composite fermions is that as they move about, the phases originating from vortices partly cancel the Aharonov Bohm phases due to an external magnetic field.

Hence composite fermions experience much reduced effective magnetic field.

$$B^* = B - 2p\rho\phi_0 \tag{19}$$

Conclusion

The following picture has finally emerged. First, the electrons form Landau because of the quantisation of their kinetic energy. This results in IQHE. Within the lowest Landau level, electrons by capturing vortices and transforming into fermions. Even though composite fermions are quantum mechanical particles with a true manybody character, they may be treated for most purposes, as ordinary non-interacting fermions moving in an effective magnetic field. They form quasi Landau levels and execute cyclotron motion. The formation of composite fermions lies at the root of the FQHE and several other fascinating experimental phenomena.