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Quantum Mechanics: Scattering theory

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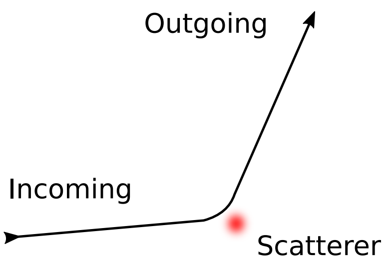
Scattering as time-dependent perturbation

Scattering is normally viewed as a particle coming from a distance and briefly interacting with a fixed *scatterer*, and the moving far away. This is pictorially depicted in the figure. One would naively imagine treating a localized particle moving in time, briefly coming clode to the scatterer, and then moving away. However, in quantum mechanics the incoming particle is treated as a plane wave, which in effect means that the particle is spread out over all positions, and the scattering potential is switched on at a specific time. One may want to consider the particle long after this time. Before interacting with the scatterer, the particle is assumed to be in a stationary state of the "free" Hamiltonian, and after the scatter too, it is assumed to be found in one of the eigenstates of the free Hamiltonian. So, the problem essentially becomes that of a time-dependent perturbation, and it's Hamiltonian can be written as



$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{V}(r), \tag{1}$$

where $\boldsymbol{H}_0$ most commonly is the Hamiltonian of a free particle, i.e., $\boldsymbol{H}_0 = \boldsymbol{p}^2/2m$

Guessing a solution

The equation one has to solve then, is

$$(\boldsymbol{H}_0 + \boldsymbol{V}(r))\,|\psi\rangle = E|\psi\rangle, \tag{2}$$

where E is an eigenvalue of the full Hamiltonian. The above equation can be manipulated to put in the following form:

$$(E - \boldsymbol{H}_0)\,|\psi\rangle = \boldsymbol{V}(r)|\psi\rangle, \tag{3}$$

so that we can write a *formal* solution as

$$|\psi\rangle = \frac{1}{E - \boldsymbol{H}_0}\boldsymbol{V}(r)|\psi\rangle. \tag{4}$$

It is quite obvious that fraction is in a danger of accidentally becoming singular. We can remedy that by adding an infinitesimal complex number to the denominator:

$$|\psi\rangle = \frac{1}{E - \boldsymbol{H}_0 + i\epsilon}\boldsymbol{V}(r)|\psi\rangle. \tag{5}$$

Now, if we know that before the perturbation $\boldsymbol{V}(r)$ is applied, the system is in an "initial" eigenstate of $\boldsymbol{H}_0$, say, $i\rangle$, we can add it by hand,

$$|\psi\rangle = |i\rangle + \frac{1}{E - \boldsymbol{H}_0 + i\epsilon}\boldsymbol{V}(r)|\psi\rangle, \tag{6}$$

becuase when $V(r) = 0$, the state is $|\psi\rangle = |i\rangle$. Equation (6) is known as the **Lippmann-Schwinger equation**. That, of course, was a very ad-hoc way getting to it, although it turns out to be the right equation in the end.

The interaction picture

We will first recall the interaction picture, which is very useful in studying time-dependent perturbation theory. If the system is governed by a Hamiltonian which has the following form:

$$H = H_0 + V(t), \tag{7}$$

where $V(t)$ is a time-dependent perturbing potential. We can define a state and operator in the interaction picture as follows:

$$|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle \qquad A_I = e^{iH_0t}Ae^{-iH_0t} \tag{8}$$

One can write a Schrödinger-like differential equation for $|\psi(t)\rangle_I$ from (8)

$$i\hbar\frac{\partial|\psi(t)\rangle_I}{\partial t} = V_I|\psi(t)\rangle_I. \tag{9}$$

A formal solution to the above would look like

$$|\psi(t)\rangle_I = U_I(t)|\psi(0)\rangle_I. \tag{10}$$

Time-dependent perturbation theory

Here we will follow a more rigorous approach to studying the problem of scattering. Suppose after the scattering, at time t the system is in a state $|\psi(t)\rangle$. We seek the probability that it is found in the eigenstate of the unperturbed Hamiltonian, $|n(t) = e^{-iH_0t}|n\rangle$. The probability amplitude is given by

$$\begin{aligned}\langle n(t)|\psi(t)\rangle &= \langle n|e^{iH_0t}|\psi(t)\rangle \\ &= \langle n|\psi(t)\rangle_I\end{aligned} \tag{11}$$

Using (9) we can write

$$\begin{aligned}\int_{t_0}^{t}\frac{\partial|\psi(t')\rangle_I}{\partial t'}dt' &= -\frac{i}{\hbar}\int_{t_0}^{t}V_I|\psi(t')\rangle_I dt' \\ |\psi(t)\rangle_I - |\psi(0)\rangle_I &= -\frac{i}{\hbar}\int_{t_0}^{t}V_I|\psi(t')\rangle_I dt' \\ |\psi(t)\rangle_I &= |\psi(0)\rangle_I - \frac{i}{\hbar}\int_{t_0}^{t}V_I|\psi(t')\rangle_I dt'\end{aligned} \tag{12}$$

With all the formalism in place, we now turn back to our problem of scattering where the particle is assumed to be in an initial state $|i\rangle$, such that $H_0|i\rangle = E_i|i\rangle$. We assume that at time t_0 the perturbing potential $V(r)$ is switched on. The probability amplitude of finding the system in a state $|n(t)\rangle$, such that $H_0|n(t)\rangle = E_i|n(t)\rangle$, is given by

$$\langle n|\psi(t)\rangle_I = \langle n|i\rangle - \frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I|\psi(t')\rangle_I dt' \tag{13}$$

The state of the system at time t can be written in terms of the state at time $t = 0$, as follows:

$$|\psi(t)\rangle_I = \boldsymbol{U}_I(t,t_0)|\psi(0)\rangle_I = \boldsymbol{U}_I(t,t_0)|i\rangle, \tag{14}$$

which can transform our equation of interest as

$$\begin{aligned}
\langle n|\boldsymbol{U}_I(t,t_0)|i\rangle &= \langle n|i\rangle - \frac{i}{\hbar}\int_{t_0}^{t}\langle n|\boldsymbol{V}_I\boldsymbol{U}_I(t',t_0)|i\rangle dt' \\
&= \langle n|i\rangle - \sum_m \frac{i}{\hbar}\int_{t_0}^{t}\langle n|\boldsymbol{V}_I|m\rangle\langle m|\boldsymbol{U}_I(t',t_0)|i\rangle dt' \\
&= \langle n|i\rangle - \sum_m \frac{i}{\hbar}\int_{t_0}^{t}\langle n|e^{i\boldsymbol{H}_0 t'}\boldsymbol{V}e^{-i\boldsymbol{H}_0 t'}|m\rangle\langle m|\boldsymbol{U}_I(t',t_0)|i\rangle dt' \\
&= \langle n|i\rangle - \sum_m \langle n|\boldsymbol{V}|m\rangle\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{nm}t'}\langle m|\boldsymbol{U}_I(t',t_0)|i\rangle dt',
\end{aligned} \tag{15}$$

where $\omega_{nm} = (E_n - E_m)/\hbar$. At this stage we would like to recall the transition amplitude that is calculated in the first-order perturbation theory, which leads us to the very useful *Fermi golden rule*. That transition amplitude is given by

$$\langle n|\boldsymbol{U}_I(t,t_0)|i\rangle = \langle n|i\rangle - \langle n|\boldsymbol{V}|i\rangle\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}dt'. \tag{16}$$

We insist that we would like to put (15) in the form (16). We conjecture the following form for (15):

$$\langle n|\boldsymbol{U}_I(t,t_0)|i\rangle = \langle n|i\rangle - T_{ni}\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}dt', \tag{17}$$

where T_{ni} are certain matrix elements which are supposed to play the same role that $V_{ni} \equiv \langle n|\boldsymbol{V}|i\rangle$ play in the Fermi golden rule. Right now we do not know what T_{ni} are, but we will seek a solution for them. We may also be interesting in looking at our system long after the scattering has occured, which means t_0 could be long back in the past, close to $-\infty$. However, the integrand in the above being oscillating, it is difficult to find a convergent solution. We remedy this problem by adding a small term $\zeta t'$ in the exponent. This will make the integrand go to zero as $t_0 \longrightarrow -\infty$. Of course in the end we will put $\zeta = 0$. At the current time, $\zeta t \sim 0$. Thus the above equation can be recast as

$$\begin{aligned}
\langle n|\boldsymbol{U}_I(t,t_0)|i\rangle &= \langle n|i\rangle - T_{ni}\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'+\zeta t'}dt' \\
&= \langle n|i\rangle - T_{ni}\,\frac{i}{\hbar}\frac{e^{i\omega_{ni}t'+\zeta t'}}{i\omega_{ni}+\zeta}\bigg|_{-\infty}^{t} \\
&= \langle n|i\rangle + T_{ni}\frac{e^{i\omega_{ni}t}}{-\hbar\omega_{ni}+i\hbar\zeta}
\end{aligned} \tag{18}$$

In the following we use $\epsilon \equiv \hbar\zeta$. We subsitute $\boldsymbol{U}_I(t',t_0)$ in the integral on the RHS of (15), by (18), which leads us to

$$\langle n|\boldsymbol{U}_I(t,t_0)|i\rangle = \langle n|i\rangle - \sum_m \langle n|\boldsymbol{V}|m\rangle \frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{nm}t'}\langle m|i\rangle + V_{nm}T_{mi}\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{nm}t'}\frac{e^{i\omega_{mi}t}}{-\hbar\omega_{mi}+i\epsilon}dt'$$
$$= \langle n|i\rangle - \langle n|\boldsymbol{V}|i\rangle\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}dt' + \sum_m V_{nm}\frac{T_{mi}}{-\hbar\omega_{mi}+i\epsilon}\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}dt'$$
$$= \langle n|i\rangle - \left(\langle n|\boldsymbol{V}|i\rangle + \sum_m \langle n|\boldsymbol{V}|m\rangle\frac{T_{mi}}{-\hbar\omega_{mi}+i\epsilon}\right)\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}dt' \tag{19}$$

Comparing the above with (17), we conclude that

$$T_{ni} = \langle n|\boldsymbol{V}|i\rangle + \sum_m \langle n|\boldsymbol{V}|m\rangle\frac{T_{mi}}{-\hbar\omega_{mi}+i\epsilon} \tag{20}$$

This is a system of linear equations, with as many equations as the number of unknowns. Solution would mean T_{ni} will be expressible as a linear combination of V_{nm}:

$$T_{ni} = \sum_m \langle n|\boldsymbol{V}|m\rangle c_{nm}, \tag{21}$$

where c_{nm} are certain constants. We conjecture that there is a state $|\psi_i^+\rangle$ such that c_{nm} are its expansion coeeficients in the basis $\{|m\rangle\}$, i.e., $|\psi_i^+\rangle = \sum_j c_{nj}|j\rangle$. Thus $c_{nm} = \langle m|\psi_i^+\rangle$. The above equation can then be written as

$$T_{ni} = \langle n|\boldsymbol{V}\sum_m |m\rangle c_{nm} = \langle n|\boldsymbol{V}|\psi_i^+\rangle. \tag{22}$$

There is a label i on $|\psi_i^+\rangle$ because c_{nm} will be different for each T_{ni}. Equation (20) can then be written as

$$\langle n|\boldsymbol{V}|\psi_i^+\rangle = \langle n|\boldsymbol{V}|i\rangle + \sum_m \langle n|\boldsymbol{V}|m\rangle\frac{1}{-\hbar\omega_{mi}+i\epsilon}\langle m|\boldsymbol{V}|\psi_i^+\rangle$$
$$= \langle n|\boldsymbol{V}|i\rangle + \sum_m \langle n|\boldsymbol{V}\frac{1}{E_i - E_m + i\epsilon}|m\rangle\langle m|\boldsymbol{V}|\psi_i^+\rangle$$
$$= \langle n|\boldsymbol{V}|i\rangle + \sum_m \langle n|\boldsymbol{V}\frac{1}{E_i - H_0 + i\epsilon}|m\rangle\langle m|\boldsymbol{V}|\psi_i^+\rangle$$
$$= \langle n|\boldsymbol{V}|i\rangle + \langle n|\boldsymbol{V}\frac{1}{E_i - H_0 + i\epsilon}\boldsymbol{V}|\psi_i^+\rangle \tag{23}$$

Since the above must be true for all $|n\rangle$

$$\boldsymbol{V}|\psi_i^+\rangle = \boldsymbol{V}|i\rangle + \boldsymbol{V}\frac{1}{E_i - H_0 + i\epsilon}\boldsymbol{V}|\psi_i^+\rangle, \tag{24}$$

or

$$|\psi_i^+\rangle = |i\rangle + \frac{1}{E_i - H_0 + i\epsilon}\boldsymbol{V}|\psi_i^+\rangle. \tag{25}$$

This is the Lippmann-Schwinger equation, guessed earlier in (6). So we see that the Lippmann-Schwinger equation comes out by treating scattering as a time-dependent perturbation. Wave-function is straight-forward to calculate from the above:

$$\begin{aligned}\psi_i^+(\boldsymbol{r}) &= \langle \boldsymbol{r}|\psi_i^+\rangle = \langle \boldsymbol{r}|i\rangle + \langle \boldsymbol{r}|\frac{1}{E_i - H_0 + i\epsilon}\boldsymbol{V}|\psi_i^+\rangle \\ &= \langle \boldsymbol{r}|i\rangle + \int d\boldsymbol{r}'\langle \boldsymbol{r}|\frac{1}{E_i - H_0 + i\epsilon}|\boldsymbol{r}'\rangle\langle \boldsymbol{r}'|\boldsymbol{V}\psi_i^+\rangle. \end{aligned} \tag{26}$$

Assuming that we are dealing with free particles coming and getting scattered, the unperturbed Hamiltonian $\boldsymbol{H}_0 = \frac{1}{2m}\boldsymbol{p}^2$, such that $\boldsymbol{H}_0|\boldsymbol{k}\rangle = \frac{\hbar^2 k^2}{2m}|\boldsymbol{k}\rangle$. We introduce a complete set of momentum states ($\sum |\boldsymbol{k}'\rangle\langle \boldsymbol{k}'| \rightarrow \frac{1}{(2\pi)^3}\int d\boldsymbol{k}'|\boldsymbol{k}'\rangle\langle \boldsymbol{k}'|$) in the above to get

$$\begin{aligned}\psi_i^+(\boldsymbol{r}) &= \langle \boldsymbol{r}|i\rangle + \int d\boldsymbol{r}'\frac{1}{(2\pi)^3}\int d\boldsymbol{k}'\langle \boldsymbol{r}|\frac{1}{E_i - H_0 + i\epsilon}|\boldsymbol{k}'\rangle\langle \boldsymbol{k}'|\boldsymbol{r}'\rangle\langle \boldsymbol{r}'|\boldsymbol{V}\psi_i^+\rangle \\ &= \langle \boldsymbol{r}|i\rangle + \int d\boldsymbol{r}'\frac{1}{(2\pi)^3}\int d\boldsymbol{k}'\langle \boldsymbol{r}|\boldsymbol{k}'\rangle\frac{1}{\hbar^2 k^2/2m - \hbar^2 k'^2/2m + i\epsilon}\langle \boldsymbol{k}'|\boldsymbol{r}'\rangle\langle \boldsymbol{r}'|\boldsymbol{V}\psi_i^+\rangle \\ &= \langle \boldsymbol{r}|i\rangle + \int d\boldsymbol{r}'\frac{1}{(2\pi)^3}\int d\boldsymbol{k}'\frac{e^{i(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{k}'}}{\hbar^2 k^2/2m - \hbar^2 k'^2/2m + i\epsilon}\langle \boldsymbol{r}'|\boldsymbol{V}\psi_i^+\rangle \\ &= \langle \boldsymbol{r}|i\rangle + \int d\boldsymbol{r}' G_+(\boldsymbol{r},\boldsymbol{r}')\langle \boldsymbol{r}'|\boldsymbol{V}\psi_i^+\rangle, \end{aligned} \tag{27}$$

where $G_+(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}'\frac{e^{i(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{k}'}}{k^2 - k'^2 + i\epsilon'}$ is called the Green's function.

The Scattered Wavefunction

Wave-function is straight-forward to calculate from the above: Assuming that we are dealing with free particles coming and getting scattered, the unperturbed Hamiltonian $\boldsymbol{H}_0 = \frac{1}{2m}\boldsymbol{p}^2$, such that $\boldsymbol{H}_0|k\rangle = \frac{\hbar^2 k^2}{2m}|k\rangle$.

$$\begin{aligned}\langle \boldsymbol{r}|\psi^+\rangle &= \langle \boldsymbol{r}|i\rangle + \frac{2m}{\hbar^2}\int d\boldsymbol{r}' G_+(\boldsymbol{r},\boldsymbol{r}')\langle \boldsymbol{r}'|\boldsymbol{V}\psi^+\rangle \\ \psi^+(\boldsymbol{r}) &= \psi_{inc}(\boldsymbol{r}) + \frac{2m}{\hbar^2}\int d\boldsymbol{r}' G_+(\boldsymbol{r},\boldsymbol{r}')V(\boldsymbol{r}')\psi^+(\boldsymbol{r}'), \end{aligned} \tag{28}$$

where $G_+(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}'\frac{e^{i(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{k}'}}{k^2 - k'^2 + i\epsilon'}$ is called the Green's function.

The 3-dimensional integral over momentum can be represented in spherical polar coordinates as, $\int d\boldsymbol{k}' = \int k'^2 dk' \int_0^{\pi}\sin\theta' d\theta' \int_0^{2\pi} d\phi'$. Additionally, the momentum coordinates can be so chosen that θ' is the angle between $\boldsymbol{k}'$ and $\boldsymbol{r}-\boldsymbol{r}'$, so that $(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{k}' =$

$|\boldsymbol{r}-\boldsymbol{r}'|k\cos\theta'$. Thus the Green's function has the form

$$\begin{aligned} G_+(\boldsymbol{r},\boldsymbol{r}') &= \frac{2\pi}{(2\pi)^3}\int_0^\infty k'^2 dk' \int_0^\pi \frac{e^{i|\boldsymbol{r}-\boldsymbol{r}'|k'\cos\theta'}}{k^2-k'^2+i\epsilon'}\sin\theta' d\theta' \\ &= \frac{1}{(2\pi)^2}\int_0^\infty k'^2 dk' \int_{-1}^1 \frac{e^{i|\boldsymbol{r}-\boldsymbol{r}'|k'\mu}}{k^2-k'^2+i\epsilon'}d\mu \\ &= \frac{1}{(2\pi)^2}\int_0^\infty \frac{k'^2}{k^2-k'^2+i\epsilon'}\frac{e^{i|\boldsymbol{r}-\boldsymbol{r}'|k'}-e^{-i|\boldsymbol{r}-\boldsymbol{r}'|k'}}{i|\boldsymbol{r}-\boldsymbol{r}'|k'}dk' \\ &= \frac{1}{(2\pi)^2 i|\boldsymbol{r}-\boldsymbol{r}'|}\int_{-\infty}^\infty \frac{k'e^{i|\boldsymbol{r}-\boldsymbol{r}'|k'}}{k^2-k'^2+i\epsilon'}dk', \end{aligned} \tag{29}$$

where integration variable $\mu=\cos\theta'$ was chosen. It is now time to mention the reason for putting super- and sub-scripts "+" on $|\psi^+\rangle$ and $G_+(\boldsymbol{r},\boldsymbol{r}')$. By the taking a complex conjugate of equation (28) on can also study **scattering backwards in time**. In this situation, the state will be $|\psi^-\rangle$ and the Green's function $G_-(\boldsymbol{r},\boldsymbol{r}')$ will have a $-i\epsilon'$ instead of $i\epsilon'$. In the following analysis, we will only consider scattering forward in time, and hence drop the subscript "+".

Now the integrand in $G_+(\boldsymbol{r},\boldsymbol{r}')$ has poles at $k'\approx k+i\epsilon/2k$ and $k'\approx -k-i\epsilon/2k$. The integral can be evaluated by the method of residues using contour integration. If we use a semicircular contour in the upper part of the complex plane, it will contain only one pole at $k'=k+i\epsilon/2k$. The other pole at $k'=-k-i\epsilon/2k$, lies in the lower part, outside the contour. The value of the integral will be $2\pi i\times$ residue:

$$\int_{-\infty}^\infty \frac{k'e^{i|\boldsymbol{r}-\boldsymbol{r}'|k'}}{k^2-k'^2+i\epsilon'}dk' = -\pi i e^{i|\boldsymbol{r}-\boldsymbol{r}'|k} \tag{30}$$

The Green's function now simplifies to

$$G_+(\boldsymbol{r},\boldsymbol{r}') = -\frac{e^{i|\boldsymbol{r}-\boldsymbol{r}'|k}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{31}$$

The expression for the scattered wave also simplifies to

$$\psi(\boldsymbol{r}) = \psi_{inc}(\boldsymbol{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{i|\boldsymbol{r}-\boldsymbol{r}'|k}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi(\boldsymbol{r}')d\boldsymbol{r}'. \tag{32}$$

Asymptotic limit

Now that we have a general expression for scatter wavefunction, we consider the realistic situation where the detector (at position $\boldsymbol{r}$) is kept far away from the scatterer, in the sense that the range of the scattering potential is short compared to $\boldsymbol{r}$. The integral over $\boldsymbol{r}'$ will effectively be confined to only those value of $\boldsymbol{r}'$ where the potential is nonzero. So, for a detector kept far away from the scatterer, we have the situation $r \ll r'$ or $r/r' \ll 1$. So, we can use the following approximations

$$e^{i|\boldsymbol{r}-\boldsymbol{r}'|k} = e^{ik\sqrt{(\boldsymbol{r}-\boldsymbol{r}')^2}} = e^{ik\sqrt{r^2-2\boldsymbol{r}\cdot\boldsymbol{r}'+r'^2}} = e^{ikr\sqrt{1-2\boldsymbol{r}\cdot\boldsymbol{r}'/r^2+(r'/r)^2}} \approx e^{ikr-k(\boldsymbol{r}/r)\cdot\boldsymbol{r}'} = e^{ikr-\boldsymbol{k}\cdot\boldsymbol{r}'}$$

and $\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r}$. The asymptotic expression for the scattered wave assumes the form

$$\psi(\mathbf{r}) = \psi_{inc}(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'. \tag{33}$$

It should be mentioned here that the integral in the above equation no longer depends on r. It only depends on the angles θ, ϕ corresponding to $\mathbf{r}$.

Scattering Amplitude and Cross Section

If the incoming wavefunction is a plane wave, say $Ae^{i\mathbf{k}_0\cdot\mathbf{r}}$, (41) has the form

$$\psi(\mathbf{r}) = A\left[e^{i\mathbf{k}_0\cdot\mathbf{r}} - \frac{e^{ikr}}{r}f(\theta,\phi)\right], \tag{34}$$

where $f(\theta,\phi) = \frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{k}\cdot\mathbf{r}'}V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'$. In (34) the first term on RHS represents the incoming plane wave, and the second term represents an outgoing spherical wave e^{ikr}/r, times an amplitude $f(\theta,\phi)$. Thus $f(\theta,\phi)$ can be interpreted as the scattering amplitude.

The number of particles scattered into a solid angle element $d\Omega$, which is just short for $\sin\theta d\theta d\phi$, is proportional to an important quantity, the differential cross section. The differential cross section, represented by $\frac{d\sigma}{d\Omega}$, is defined as the number of particles scattered into an element of solid angle $d\Omega$ in the direction given by θ, ϕ, per unit time, per unit incident flux:

$$\frac{d\sigma}{d\Omega} = \frac{1}{J_{inc}}\frac{dN(\theta,\phi)}{d\Omega}, \tag{35}$$

where J_{inc} is the incoming flux of particles. For a given state $\psi(\mathbf{r})$ the flux density is defined as

$$\mathbf{J} = \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) \tag{36}$$

Using the above definition, the magnitude of incoming flux density is given by

$$J_{inc} = |A|^2\frac{\hbar k_0}{2m}, \tag{37}$$

and the magnitude of scatterd flux density, in the direction θ, ϕ, is

$$J_{scr} = |A|^2\frac{\hbar k}{2mr^2}|f(\theta,\phi)|^2. \tag{38}$$

The number of particles scattered into a solid angle $d\Omega$, and passing through an area element $dA = r^2 d\Omega$, per unit time, is given by

$$dN(\theta,\phi) = J_{scr}r^2 d\Omega = |A|^2\frac{\hbar k}{2m}|f(\theta,\phi)|^2 d\Omega. \tag{39}$$

The differential cross-section then turns out to be

$$\frac{d\sigma}{d\Omega} = \frac{1}{J_{inc}}\frac{dN(\theta,\phi)}{d\Omega} = \frac{k}{k_0}|f(\theta,\phi)|^2. \tag{40}$$

Born approximation

Although we have obtained expressions for the scattering amplitude and the differential scattering cross-section, we cannot calculate them yet, as $f(\theta,\phi)$ involves the scattered state $\psi(\boldsymbol{r})$, which we do not know yet. We can make an approximation by replacing the state $\psi(\boldsymbol{r})$ in $f(\theta,\phi)$ by the incoming state $\psi_{inc}(\boldsymbol{r})$. This is called the first Born approximation.

$$\begin{aligned}\psi(\boldsymbol{r}) &\approx \psi_{inc}(\boldsymbol{r}) - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'}V(\boldsymbol{r}')\psi_{inc}(\boldsymbol{r}')d\boldsymbol{r}' \\ &= Ae^{i\boldsymbol{k_0}\cdot\boldsymbol{r}} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{i(\boldsymbol{k_0}-\boldsymbol{k})\cdot\boldsymbol{r}'}V(\boldsymbol{r}')d\boldsymbol{r}'. \end{aligned} \quad (41)$$

If the scattering is elastic, $k = k_0$, although $\boldsymbol{k} \neq \boldsymbol{k_0}$. Also, $|\boldsymbol{k_0}-\boldsymbol{k}| = \sqrt{k_0^2 + k^2 - 2k_0 k\cos\theta} = 2k\sin(\theta/2)$. The scattering amplitude and the differential cross section can now be written as

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i(\boldsymbol{k_0}-\boldsymbol{k})\cdot\boldsymbol{r}'}V(\boldsymbol{r}')d\boldsymbol{r}' \quad (42)$$

and

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 = \frac{m^2}{4\pi^2\hbar^4}\left|\int e^{i(\boldsymbol{k_0}-\boldsymbol{k})\cdot\boldsymbol{r}'}V(\boldsymbol{r}')d\boldsymbol{r}'\right|^2 \quad (43)$$

In the 3-dimensional integral over $\boldsymbol{r}'$, integrals over the angles can be carried out to get reduced expressions for $f(\theta,\phi)$ and $\frac{d\sigma}{d\Omega}$.