Quantum Mechanics: Quantum dynamics

Schrödinger picture

In the original way in which quantum mechanics was formulated, the dynamics of the quantum state is built into the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle.$$

This equation can be solved, and its formal solution written as

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle,$$

where $|\psi(0)\rangle$ is the state at time $t = 0$. In the Schrödinger picture, the observables do not depend on time. The time evolution is governed by the *unitary* time-evolution operator given by

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar}.$$

It is also referred to as the *propagator*. In this picture the observables do not change with time, only the state does. Suppose the system is initially in an eigenstate of the Hamiltonian $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$. Then it can be easily shown that

$$|\psi(t)\rangle = e^{-iE_nt/\hbar}|\phi_n\rangle.$$

In this situation, the state remains virtually unchanged as time progresses. The only change is in the phase factor of the state $e^{-iE_n/\hbar}$, which is not of much consequence. Hence the eigenstates of the Hamiltonian are called *stationary states*.

Heisenberg picture

Heisenberg had a different view of the dynamics of quantum systems. He believed that the observables are what evolve in time, and the states remain unchanged with time. This picture is diametrically opposite to the Schrödinger picture. An observable $\hat{A}$ has the following time dependence in the Heisenberg picture:

$$A_H(t) = e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar},$$

with $A_H(0) = \hat{A}$. The states remain unchanged with time. The equation for dynamics can be got by differentiating $A_H(t)$ with respect to t:

$$\begin{aligned}\frac{d\hat{A}_H}{dt} &= \tfrac{i}{\hbar}\hat{H}e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar} - \tfrac{i}{\hbar}\hat{H}e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar}\hat{H} \\ &= \tfrac{i}{\hbar}[\hat{H},\hat{A}_H]. \qquad (1)\end{aligned}$$

In arriving at this result, use has been made of the fact that $e^{i\hat{H}t/\hbar}$ commutes with $\hat{H}$. This fact can also be useful while evaluating the commutator in the above equation

$$\frac{d\hat{A}_H}{dt} = \tfrac{i}{\hbar}[\hat{H},\hat{A}_H] = \tfrac{i}{\hbar}e^{i\hat{H}t/\hbar}[\hat{H},\hat{A}]e^{-i\hat{H}t/\hbar},$$

so that known commutation relations can be used, without involving the time-evolution operator.

The results of measurement should not depend on what theoretical picture one uses. So, the Schrödinger picture and the Heisenberg picture should lead to the same results. For example, the time dependence of the expectation value yields the same expression in both the pictures:

$$\langle \hat{A}(t)\rangle = \langle\psi(0)|e^{i\hat{H}t/\hbar}\hat{A}e^{i\hat{H}t/\hbar}|\psi(0)\rangle,$$

$$\langle \hat{A}(t)\rangle = \langle\psi(0)|e^{i\hat{H}t/\hbar}\hat{A}e^{i\hat{H}t/\hbar}|\psi(0)\rangle.$$

Interaction (Dirac) picture

The Interaction picture, introduced by Dirac, comes in useful when the Hamiltonian consists of the original Hamiltonian plus an interaction term, and one is interested in just the effect of the interaction term on the time evolution. Consider a Hamiltonian which can be written as follows

$$\hat{H} = \hat{H}_0 + \hat{V},$$

where $\hat{V}$ is the "interaction" term. In the interaction picture, both the state and the observables evolve in time, but in a modified way. The state in the interaction picture is given by

$$|\psi(t)\rangle_I = e^{i\hat{H}_0 t/\hbar}|\psi(t)\rangle,$$

where $|\psi(t)\rangle$ is the state in the Schrödinger picture. An operator in the interaction picture is given by

$$\hat{A}_I(t) = e^{i\hat{H}_0 t/\hbar}\hat{A}e^{-i\hat{H}_0 t/\hbar},$$

where $|\psi(t)\rangle$ is the state in the Schrödinger picture. The equation of motion for $|\psi(t)\rangle_I$ can be obtained as follows:

$$\begin{aligned}
i\hbar\frac{d|\psi(t)\rangle_I}{dt} &= -e^{i\hat{H}_0 t/\hbar}\hat{H}_0|\psi(t)\rangle + e^{i\hat{H}_0 t/\hbar}(\hat{H}_0 + \hat{V})|\psi(t)\rangle \\
&= e^{i\hat{H}_0 t/\hbar}\hat{V}|\psi(t)\rangle \\
&= e^{i\hat{H}_0 t/\hbar}\hat{V}e^{-i\hat{H}_0 t/\hbar}e^{i\hat{H}_0 t/\hbar}|\psi(t)\rangle \\
&= \hat{V}_I|\psi(t)\rangle_I \qquad (2)
\end{aligned}$$

So in the interaction picture, the time evolution of the state is governed only by the interaction part of the Hamiltonian.

By an equally simple algebra, it can be shown that the time evolution of an operator, in the interaction picture, is given by

$$\frac{d\hat{A}_I}{dt} = \tfrac{i}{\hbar}[\hat{H}_0, \hat{A}_I].$$

Thus in the interaction picture, the time evolution of the observables is governed only by the "free" part of the Hamiltonian.