**Quantum Mechanics: Measurement in QM**

## Measurement in Quantum Mechanics

Quantum mechanics is a tremendously successful theory. Despite that, measurement is an aspect of quantum mechanics, which is still poorly understood. In classical mechanics a measurement implies that one extracts some information about the state the system is already in. For example, we may take a picture of a moving particle, which helps us in determining the exact position of the particle at that instant of time. It is assumed that the act of measurement has negligible effect on the state of the particle. Afterall, how much effect can taking a photograph have on the position or momentum of the particle? In the worst case, even if the act of measurement strongly disturbs the state of the system, the information is still obtained about the state of the system before it was disturbed. Strangely, in quantum mechanics none of this holds! Firstly, for a quantum system, any act of measurement completely changes its state, and the change is completely random. Secondly, the result of a single quantum measurement does not give *any* information about the previously existing state of the system.

Measurements in quantum mechanics work as follows. Let the system be in an initial state $|\psi\rangle$, and suppose that we are interested in measuring an observable $\hat{A}$. The eigenstates of the observable $\hat{A}$ are given by

$$\hat{A}|a_n\rangle = \alpha_n|a_n\rangle.$$

The state $|\psi\rangle$ of the system can be represented in terms of the states $\{|a_n\rangle\}$:

$$|\psi\rangle = \sum_n c_n|a_n\rangle,$$

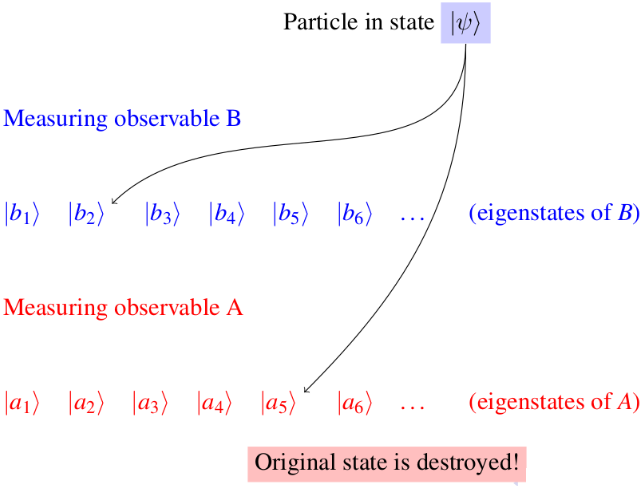
where $c_n$ are certain constants. A common misconception is that a measurement involves applying the operator ($\hat{A}$ in the present case) on the state of the system: $\hat{A}|\psi\rangle$. This is **not** correct. The dynamics of the process of measurement is not understood till now. It is still considered an unsolved problem. What we do know is that the process of measurement results in *one of the eigenvalues of* $\hat{A}$. Because of the measurement, the state of the system changes to the *eigenstate of* $\hat{A}$ corresponding to the eigenvalue which appeared. This is the third postulate quantum mechanics. Another way of stating it is that initially the system is in a superposition of various eigenstates of $\hat{A}$ ($|\psi\rangle = \sum_n c_n|a_n\rangle$), and the act of measurement *collapses* the state to one of those eigenstates.

Notice that the result one gets depends on what one probes, and the state of the system changes to a state depending on what is being probed. It does not matter what the original state was. So, the measurement yields no information about the initial state of the system. To add to the strangeness of it, which eigenvalue of $\hat{A}$ appears in a measurement, is completely random. If one repeats the same measurement by preparing the system again in the same state $|\psi\rangle$, a different eigenvalue of $\hat{A}$ may result. So, every measurement of $\hat{A}$, carried out on $|\psi\rangle$, yields a different eigenvalue. Although the process is completely random, all eigenvalues are not equally likey to occur. The probability of eigenvalue $\alpha_k$ appearing, as a result of measurent, has a probability equal to $|c_k|^2$. We have previously learnt that

$$c_k = \langle a_k|\psi\rangle.$$

So, the probability $P_k$ of the eigenvalue $\alpha_a$ occuring is

$$P_k = |\langle a_k|\psi\rangle|^2.$$




One the other hand, if one were to choose a different observable $\hat{B}$ for measurement, a similar scenario would occur. If the eigenstate of $\hat{B}$ are given by

$$\hat{B}|b_n\rangle = \beta_n|b_n\rangle,$$

the state $|\psi\rangle$ can be represented as

$$|\psi\rangle = \sum_n d_n|b_n\rangle,$$

where $d_n = \langle b_n|\psi\rangle$ are constants. The measurement will result in a random eigenvalue of $\hat{B}$, (say) $\beta_k$, and the state of the system will change to $|b_k\rangle$. In repeated measurements, on the original state $|\psi\rangle$, eigenvalue $\beta_k$ is obtained with a probability $|\langle b_n|\psi\rangle|^2$.

There is one situation in which the state of the system does *not* change in the process of measurement. That is when the initial state of the system is an eigenstate of the observable being measured. As an example, if the state of the system was $|a_m\rangle$, the measurement will yield the eigenvalue $\alpha_m$, and the state of the system will continue to be $|a_m\rangle$. For a given initial state, if one could *choose* an operator to measure, whose one eigenstate is that initial state, one can perform a measurement without disturbing the system. For example, if a particle is in an eigenstate of momentum, making a momentum measurement will not disturb the state of the particle. However, if the initial state is *unknown*, there is no way of choosing such an operator.

## Repeated measurements

Suppose the initial state of the system is $|\psi\rangle$ and one measured the observable $\hat{A}$ and got the value $\alpha_k$, what would happen if one measures $\hat{A}$ again? Here one is not starting from the initial state $|\psi\rangle$ again, but from the state obtained after the first measurement. It is easy to see after the first measurement the state of the system will be $|a_k\rangle$, and so another measurement of $\hat{A}$ will not change the state, as it is already an eigenstate of $\hat{A}$. So, the second measurement will also yield $\alpha_k$, and so will all subsequent measurements. But suppose after the first measurement of $\hat{A}$, one measures the observable $\hat{B}$, and gets a value $\beta_j$, what if $\hat{A}$ is measured again? Will one still get $\alpha_k$? The answer is no, because after measuring $\hat{B}$, the state changes to $|b_j\rangle$, which is *not* and eigenstate of $\hat{A}$. Measuring $\hat{A}$ on the system in the state $|b_j\rangle$ will yield any random eigenvalue of $\hat{A}$, not necessarily $\alpha_k$. That is why such observables are called *incompatible* observables. Measurement of one, interferes with the measurement of the other.

However, if two observables *commute*, and all their eigenstates are common eigenstates of both, then measurement of one will not affect the measurement result of the other. In this case they are called *compatible* observables.

## Expectation value

Now we recognize the *eigenvalues* of an operator as the values which appear when one measures the observable addociated with the operator. But what about the *expectation value*? Does it have any meaning in terms of measurements? If the state of the system is $|\psi\rangle$, the expectation value of the observable $\hat{A}$ is given by

$$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle.$$

Remembering that $|\psi\rangle = \sum_n c_n|a_n\rangle$, we can rewrite the above as

$$\begin{aligned}\langle\hat{A}\rangle &= \sum_{n,m}\langle a_m|c_m^*\hat{A}c_n|a_n\rangle \\ &= \sum_{n,m} c_m^* c_n\langle a_m|\alpha_n|a_n\rangle = \sum_{n,m} c_m^* c_n \alpha_n \delta nm \\ &= \sum_n |c_n|^2\alpha_n. \end{aligned} \tag{1}$$

But $|c_n|^2$ is the probability of getting the eigenvalue $\alpha_n$. So the above expression is essentially $\langle\hat{A}\rangle = \sum_n P_n\alpha_n$, which is just the definition of the *weighted average* of the eigenvalues. The expectation value of an observable can then be interpreted as the average of the measured value of the observable over many measurements of $\hat{A}$, in the state $|\psi\rangle$.

## Meaning of inner-product in terms of measurements

Now that we understand how measurements work in quantum mechanics, we can use the concept to understand the meaning of an *inner product*, which we have been treating as only a mathematical definition. We have seen that if the state of a system is $|\psi\rangle$, and one measures an observable $\hat{A}$, the probability of getting a particular eigenstate $|a_k\rangle$ of $\hat{A}$ is $|\langle a_k|\psi\rangle|^2$. The quantity $\langle a_k|\psi\rangle$ is the the *probability amplitude* of getting $|a_k\rangle$.

This means that an arbitrary *inner product* $\langle\phi|\psi\rangle$ *can be interpreted as the probability amplitude of finding the system in the state* $|\phi\rangle$*, given that it was initially in the state* $|\psi\rangle$. This, of course, assumes that a measurement of an observable is made, which $|\phi\rangle$ is an eigenstate of.