Quantum Mechanics: Angular momentum

Construction of angular momentum operators

Quantum mechanical Hamiltonian was constructed from the classical Hamiltonian simply by replacing the position and momenta by their respective operators. We shall follow the same method for constructing the angular momentum operators. Classically the angular momentum is given by

$$\vec{L} = \vec{R} \times \vec{P},$$

where $\vec{R}, \vec{P}$ represent the position and momentum vector for the particle. Now we define the quantum angular momentum operator as

$$\vec{\hat{L}} = \vec{\hat{R}} \times \vec{\hat{P}},$$

where the vector angular momentum operator is given by $\vec{\hat{L}} = \hat{i}\hat{L}_x + \hat{j}\hat{L}_y + \hat{k}\hat{L}_z$, and $\hat{L}_x, \hat{L}_y, \hat{L}_z$ are the operators for the three angular momentum components. A little algebra gives the following form of the angular momentum operators in terms of position and momentum operators:

$$\begin{aligned}\hat{L}_x =& \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y,\\ \hat{L}_y =& \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z,\\ \hat{L}_z =& \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x.\end{aligned} \tag{1}$$

Another operator of importance is the *total angular momentum* operator

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2.$$

All of the above are Hermitian operators. The commutation relations of these operators can be derived from the above relations. An example is presented here:

$$\begin{aligned}[\hat{L}_x, \hat{L}_y] =& [\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y, \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z]\\ =& [\hat{Y}\hat{P}_z, \hat{Z}\hat{P}_x] - [\hat{Y}\hat{P}_z, \hat{X}\hat{P}_z] - [\hat{Z}\hat{P}_y, \hat{Z}\hat{P}_x] + [\hat{Z}\\ =& \hat{Y}[\hat{P}_z, \hat{Z}]\hat{P}_x + \hat{X}[\hat{Z}, \hat{P}_z]\hat{P}_y = i\hbar(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x)\\ =& i\hbar\hat{L}_z.\end{aligned}$$

Commutation relation between the other operators can be similarly derived, and the result is

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y. \tag{3}$$

Interestingly, although the total angular momentum operator $\hat{L}^2$ depends on all the three components, which do not commute with each other, $\hat{L}^2$ itself commutes with all the three components:

$$[\hat{L}^2, \hat{L}_x] = 0, \quad [\hat{L}^2, \hat{L}_y] = 0, \quad [\hat{L}^2, \hat{L}_z] = 0. \qquad (4)$$

Also interesting to notice is that the although cross product of any vector with itself is always zero, that is not true for the vector angular momentum operator $\vec{\hat{L}}$. The following result is left as an excercise for the reader:

$$\vec{\hat{L}} \times \vec{\hat{L}} = i\hbar\vec{\hat{L}}.$$

Eigenvalues and eigenstates of angular momentum operators

The next step will be to find out out what values the angular momentum, and its components can take. Thus we need to find the eigenvalues and eigenstates of these operators. In the methodology followed in the undergraduate courses in quantum mechanics, one first writes the angular momentum operators in the position representation, using spherical polar coordinates. The differential equations thus obtained, are solved to get the eigenfunctions which are called *spherical harmonics*. The eigenvalues also emerge in the process. As we had argued earlier, most problems in quantum mechanics can be solved using the just the commutation properties of operators, and their actions on various states. We shall take this approach to find the eigenvalues of the angular momentum operators.

It is convenient to introduce the following *non-Hermitian* operators:

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y.$$

It is trivial to see that $[\hat{L}_\pm, \hat{L}^2] = 0$. More useful are the following relations:

$$[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z, \quad [\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm.$$

Now the fact that $[\hat{L}^2, \hat{L}_z] = 0]$, means that $\hat{L}^2$ and $\hat{L}_z$ have a common set of eigenstates. As a minimal condition, the eigenstates should depend on the eigenvalues of $\hat{L}^2$ and $\hat{L}_z$. Eigenvalues of $\hat{L}^2$ and $\hat{L}_z$ need not be the same. Keeping this in mind let us denote the eigenstates and the eigenvaues as follows:

$$\hat{L}_z|\alpha,\beta\rangle = \beta|\alpha,\beta\rangle, \quad \hat{L}^2|\alpha,\beta\rangle = \alpha|\alpha,\beta\rangle.$$

Now $\hat{L}_\pm|\alpha,\beta\rangle$ should be just another state. Let us check if this state is also an eigenstate of $\hat{L}_z$ and $\hat{L}^2$.

$$\begin{aligned}\hat{L}_z(\hat{L}_\pm|\alpha,\beta\rangle) &= (\hat{L}_\pm\hat{L}_z \pm \hbar\hat{L}_\pm)|\alpha,\beta\rangle \\ &= (\hat{L}_\pm\beta|\alpha,\beta\rangle \pm \hbar\hat{L}_\pm)|\alpha,\beta\rangle \\ &= (\beta \pm \hbar)\hat{L}_\pm|\alpha,\beta\rangle. \end{aligned} \tag{5}$$

This means, $\hat{L}_+|\alpha,\beta\rangle$ is also an eigenstate of $\hat{L}_z$, with an eigenvalue $\beta+\hbar$, and $\hat{L}_-|\alpha,\beta\rangle$ is an eigenstate of $\hat{L}_z$, with an eigenvalue $\beta-\hbar$. This effect is irrespective of what β is. This naturally implies (which can be verified):

$$\hat{L}_z(\hat{L}_\pm^n|\alpha,\beta\rangle) = (\beta \pm n\hbar)\hat{L}_\pm|\alpha,\beta\rangle.$$

So, the effect of $\hat{L}_+$ is to increase the eigenvalue of $\hat{L}_z$ by $\hbar$, and the effect of $\hat{L}_-$ is to decrease it by $\hbar$. For this reason $\hat{L}_\pm$ are called *ladder operators*. Interestingly

$$\hat{L}^2\hat{L}_\pm|\alpha,\beta\rangle = \alpha\hat{L}_\pm|\alpha,\beta\rangle,$$

which means, in the new eigenstates $\hat{L}^2$ has the same eigenvalue as it has in $|\alpha,\beta\rangle$. In other words, the various eigenstates generated by the multiple action of $\hat{L}_\pm$ are all *degenerate*, as far as $\hat{L}^2$ is concerned.

Since the eigenvalue of $\hat{L}_z$ increases and decreases in the steps of $\hbar$, so it must be a multiple of $\hbar$. So we can make the following notational change

$$\beta = m\hbar,$$

where m is an unknown number. So our modified eigenvalue relations are

$$\hat{L}_z|\alpha,m\rangle = m\hbar|\alpha,m\rangle, \quad \hat{L}^2|\alpha,m\rangle = \alpha|\alpha,m\rangle.$$

The states $|\alpha, m\rangle$ are identical to $|\alpha, \beta\rangle$, but only the label has changed. The action of $\hat{L}_\pm$ on $|\alpha, m\rangle$ increases or decreases m by 1. But since $m\hbar$ denotes the value of the z-component of the angular momentum, it must have a maximum and a minimum value. Let us denote the maximum and minimum values of m by m_{max} and m_{min}, respectively. Then by definition, the action of $\hat{L}_+$ on $|\alpha, m_{max}\rangle$ should not lead to a new higher eigenstate. Similarly, the action of $\hat{L}_-$ on $|\alpha, m_{min}\rangle$ should not lead to a new lower eigenstate. Thus we should have

$$\hat{L}_+|\alpha, m_{max}\rangle = 0, \qquad \hat{L}_-|\alpha, m_{min}\rangle = 0. \tag{6}$$

We also keep in mind that all these states are degenerate for $\hat{L}^2$:

$$\hat{L}^2|\alpha, m_{max}\rangle = \alpha|\alpha, m_{max}\rangle,$$

$$\hat{L}^2|\alpha, m_{min}\rangle = \alpha|\alpha, m_{min}\rangle.$$

We will use the following to relations:

$$\hat{L}_+\hat{L}_- = \hat{L}_x^2 + \hat{L}_y^2 + \hbar\hat{L}_z = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z,$$

$$\hat{L}_-\hat{L}_+ = \hat{L}_x^2 + \hat{L}_y^2 - \hbar\hat{L}_z = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z.$$

The useful aspect of the above form is that all the three operators on the RHS have common eigenstates

$|\alpha, m\rangle$. Taking the first equation of (6), and multiplying both sides by $\hat{L}_-$, we get

$$\begin{aligned}\hat{L}_-\hat{L}_+|\alpha, m_{max}\rangle &= 0\\ (\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z)|\alpha, m_{max}\rangle &= 0\\ (\alpha - \hbar^2 m^2 - \hbar^2 m)|\alpha, m_{max}\rangle &= 0. \end{aligned} \tag{7}$$

Since the state $|\alpha, m_{max}\rangle$ itself cannot be zero, the only way the above is possible is when the factor multiplying it is zero:

$$\alpha = \hbar^2 m_{max}(m_{max} + 1).$$

Using the same with the second equation of (6), we get

$$\alpha = \hbar^2 m_{min}(m_{min} - 1).$$

We have already seen that the eigenvalue of $\hat{L}^2$, α does not depend on m, so the RHS of the two equations above must be equal. This is only possible if $m_{min} = -m_{max}$. Let us give this maximum value of m a simpler name: $l \equiv m_{max}$. So we have

$$m_{max} = l, \quad m_{min} = -l, \quad \alpha = \hbar^2 l(l+1).$$

So the picture of the eigenvalues of $\hat{L}_z$, i.e., $\hbar m$, is clear now. The lowest value of m is $-l$ and it increases in the steps of 1, until it reaches $+l$. So, there are

$2l+1$ possible values of m. And for all these states, the eigenvalue of $\hat{L}^2$ is $\hbar^2 l(l+1)$.

What about the possible values of l itself? For the values of m to go from $-l$ to $+l$, in the steps of 1, l is obviously an integer. However, interestingly half-integer values of l also allow this possibility! For example, if $l = 3/2$, the following values of m are allowed:

$$m = -3/2, \quad -1/2, \quad +1/2, \quad +3/2 \ .$$

For *orbital* angular momentum, only integer values of l are allowed. We will see later that half integer values are allowed in a different kind of angular momentum, the *spin* angular momentum. Now we can label our eigenstates by l and m

$$\boxed{\hat{L}_z|l,m\rangle = \hbar m|l,m\rangle, \quad \hat{L}^2|l,m\rangle = \hbar^2 l(l+1)|l,m\rangle.}$$

So, we see that the values of the z-component of the angular momentum are *quantized*. That is what was seen in the Stern-Gerlach experiment.

We have seen that

$$\hat{L}_z\hat{L}_\pm|l,m\rangle = \hbar(m\pm 1)\hat{L}_\pm|l,m\rangle.$$

But we also know that

$$\hat{L}_z|l,m+1\rangle = \hbar(m+1)|l,m+1\rangle.$$

Does that mean $\hat{L}_\pm|l,m\rangle = |l,m\pm 1\rangle$? Well, we don't know if there is a multiplicative constant factor in addition. So what is implied by the above relations is:

$$\hat{L}_\pm|l,m\rangle = C_\pm|l,m\pm 1\rangle,$$

where the constant factors $C_\pm$ have to be determined. We take the norm of both sides, and get

$$\begin{aligned}
\langle l,m|\hat{L}_\pm^\dagger \hat{L}_\pm|l,m\rangle &= |C_\pm|^2\langle l,m|l,m\pm 1\rangle \\
\langle l,m|\hat{L}_\mp \hat{L}_\pm|l,m\rangle &= |C_\pm|^2 \\
\langle l,m|\hat{L}^2 - \hat{L}_z^2 \mp \hat{L}_z|l,m\rangle &= |C_\pm|^2 \\
C_\pm &= \hbar\sqrt{l(l+1) - m(m\pm 1)}
\end{aligned} \tag{8}$$

Thus our final relation specifying the action of $\hat{L}_\pm$ on the state $|l,m\rangle$ is

$$\hat{L}_\pm|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}|l,m\pm 1\rangle. \tag{9}$$

Eigenvalues of $\hat{L}_x$ & $\hat{L}_y$

The commutation relations between $\hat{L}^2$ and $\hat{L}_x$ are the same as those between $\hat{L}^2$ and $\hat{L}_z$. So, one carry out a similar process for $\hat{L}_x$, and the eigenvalues will come out to be identical to those of $\hat{L}_z$, but the eigenstates will be different. What one gets is the following:

$$\hat{L}_x|l,m\rangle_x = \hbar m|l,m\rangle_x, \quad \hat{L}^2|l,m\rangle_x = \hbar^2 l(l+1)|l,m\rangle_x.$$

So, the states $\{|l,m\rangle_x\}$ are common eigenstates of $\hat{L}^2$ and $\hat{L}_x$, which are $2l+1$ in number, and form a complete set. Similarly one can find out the common eigenstates of $\hat{L}^2$ and $\hat{L}_y$

$$\hat{L}_y|l,m\rangle_y = \hbar m|l,m\rangle_y, \quad \hat{L}^2|l,m\rangle_y = \hbar^2 l(l+1)|l,m\rangle_y.$$

Matrix representation

Since angular momentum involves finite Hilbert space, it is often convenient to use the matrix representaion of operators. In finding the matrix representaion in the basis of eigenstates of $\hat{L}_z$, it is useful to represent $\hat{L}_x$ and $\hat{L}_y$ in terms of $\hat{L}_\pm$:

$$\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}, \quad \hat{L}_y = \frac{\hat{L}_+ - \hat{L}_-}{2i},$$

and then use the relation (9). Following such a procedure the matrix representation of angular momentum operators, for $l=1$, can be written as

$$L_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \quad L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

$$L_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}, \quad L^2 = 2\hbar^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The eigenstates are given by

$$|1,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |1,-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$