Quantum Mechanics: The Spin

Discovery of the spin

Let us have another look at the Stern-Gerlach experiment performed in 1922. A beam of silver atoms, which are paramagnetic, was passed through an inhomogeneous magnetic field. Suppose each atom has an inherent magnetic moment μ , then the force on it in an inhomogeneous magnetic field is given by

$$\vec{F} = \nabla(\vec{\mu} \cdot B,$$

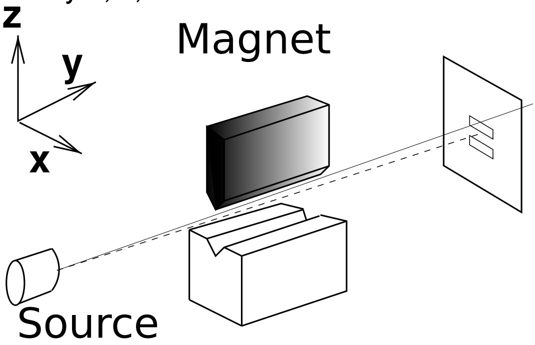
Let us assume that $B_z \gg B_x, B_y$, so that $\mu \cdot B \approx \mu_z B_z$. In this situation, the force will be primarily along the z-axis,

$$\vec{F} = \mu_z \frac{\partial B_z}{\partial z} \hat{k},$$

where $\hat{k}$ is a unit vector along z-axis. One can see that the deflection of the beam should be proportional to the z-component of the magnetic moment. The magnetic moment is related to the angular momentum by the following relation

$$\vec{\mu} = \gamma \vec{S},$$

where γ is called the gyromagnetic ratio. So the quantum observable corresponding to the z-component of the magnetic moment is $\hat{\mu}_z = \gamma \hat{S}_z$. Now we know the eigenvalues of $\hat{S}_z$ are $\hbar m$, where $m = -l, -l+1, ..., l-1, l$. For the lowest non-zero value of angular momentum, i.e. $l = 1$, there are three eigenvalues of L_z, given by $\hbar, 0, -\hbar$.



A schematic diagram of the Stern-Gerlach experiment.

Corresponding to these, there are three possible values of the magnetic moment $\mu_z = \gamma\hbar, 0, -\gamma\hbar$. These leads to three discrete values of force $F = \gamma\hbar\frac{\partial B_z}{\partial z}, 0, -\gamma\hbar$ From quantum mechanics one expects that if the silver atoms are in the ground state, then $l = 0$, and the atoms experience no force. The atomic beam should be undeflected. If the atoms are in a state with l = 1 , the atoms either experience zero force, or experience force = $\pm\gamma\hbar\frac{\partial B_z}{\partial z}$. Thus the beam should split into three, and the atoms should hit the screen at three spots. When the experiment was performed, it was observed that the beam of atoms split into only two beams. This means that the z-component of μ can have only two values! Although it confirmed the quantization of angular momentum, it went against the understanding of quantum angular momentum. It remained a puzzle for several years until in 1925 George Uhlenbeck and Samuel Goudsmit proposed an intrinsic angular momentum of the electron called "spin", which has a non-integer value $1/2$. This allowed two values of m, $\pm 1/2$, and could explain the two beams in the Stern-Gerlach experiment.

The spin angular momentum

Spin is an *intrinsic property* of elementary particles, which has nothing to do with the classical spinning motion. It is as much of an intrinsic property as mass or charge of a particle, which cannot be changed. The spin angular momentum is then described by the operators

$$\hat{S}_x, \quad \hat{S}_y, \quad \hat{S}_z, \quad \hat{S}^2,$$

which have properties identical to the orbital angular momentum operators, including the same commutation rules. Eigenstates of the operators are also similar except that they obviously do not have any spatial representation. The value of the spin is denoted by s which can take integer or half-integer values, having a $2s+1$ dimensional Hilbert space. Thus we have

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y, \tag{1}$$

and

$$[\hat{S}^2, \hat{S}_x] = 0, \qquad [\hat{S}^2, \hat{S}_y] = 0, \qquad [\hat{S}^2, \hat{S}_z] = 0. \tag{2}$$

For eigenstates of these operators, we have

$$\hat{S}_z|s,m\rangle = \hbar m|s,m\rangle, \qquad \hat{S}^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle.$$

Spin 1/2 and Qubits

Spin 1/2 is of particular importance in quantum mechanics, as the most common quantum particles, electrons, protons and neutrons, all are spin 1/2 particles. It is common to represent the eigenstates of the z-component of a spin 1/2 as

$$\hat{S}_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle, \qquad \hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle,$$

and call them “up-spin” and “down-spin” states, respectively. Matrix representation of spin 1/2 operators is very commonly used, as it is easy to manipulate 2×2 matrices. The matrix representation of the operators, in the basis of the eigenstates of $\hat{S}_z$, is given by

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The three matrices are called *Pauli matrices*, and are used more commonly than the spin matrices themselves as they do not involve the clutter of the factor $\hbar/2$. They are represented as

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

There are several properties of Pauli matrices, that are of interest, some being

$$\sigma_z^2 = \sigma_x^2 = \sigma_y^2 = \hat{1},$$

$$\{\sigma_i, \sigma_j\} = 0, \quad i \neq j.$$

The eigenstates of $\hat{S}_z$ and σ_z are given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It should be stressed here that the eigenstates of *any* two-state system can be represented by $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, irrespective of what the eigenvalues are. This aspect came in handy in dealing with *quantum bits* or *qubits* in the new field of quantum information. Classical bits are entities which can take values 0 or 1. Same way qubits are quantum observables with eigenvalues 0 and 1, with the corresponding eigenstates $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. A qubit crucially differs from a classical but by the fact that apart from having states $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, it can be in a superposition state $\begin{pmatrix}c_1\\c_2\end{pmatrix}$, with $|c_1|^2 + |c_2|^2 = 1$. There are an infinite number of such superposition states.

The commutation relations for Pauli matrices are

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k,$$

where ϵ_{ijk} is the well known antisymmetric tensor. Pauli matrices are Hermitian, and have trace zero. They all have determinant equal to -1. From the relation $\sigma_i^2 = \hat{1}$, one can easily prove

$$e^{i\alpha\sigma_i} = \hat{1}\cos\alpha + i\sigma_i \sin\alpha,$$

where α is a real number. It can be shown that *any* $2{\times}2$ matrix can be represented as a linear combination of the Pauli matrices and $\hat{1}$. As the Hilbert space is 2-dimensional, any observable of the system will be a 2×2 matrix. This means that any observable of the system can be represented in terms of the Pauli matrices.